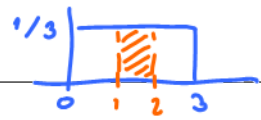
D[Nov 4]

Problem 1. (18 pt) The first column of each row in the Table below defines a new random variable X.

(a) Indicate the type of the random variable X. (There are three possible types: discrete, continuous, or mixed.) Put your answers in the second column.

(b) Find $P[1 < X \le 2]$. Put your answers in the third column. Your answers should be of the form 0._ _ _ _.

$f_X(x) = \begin{cases} \frac{1}{3}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$



	Type	$P[1 < X \le 2]$	(pt)
$X \sim$ Binomial$(3, 1/3)$	Discrete	HW	(2)
$X \sim$ Poisson(3)	Discrete	HW	(2)
$X \sim \mathcal{U}(0,3)$ (uniform on interval $(0,3)$)	Cont.	$\frac{1}{3} \approx 0.3333$	(2)
$X \sim \mathcal{E}(3)$ exponential	Cont.	0.0473	(2)
$X \sim \mathcal{N}(0,1)$ Gaussian, Normal	Cont.	$\Phi(2) - \Phi(1) \approx 0.9725 - 0.8413 \approx 0.1359$	(2)
$X \sim \mathcal{N}(1,3)$ → $\sigma_X^2 = 3$, $\sigma_X = \sqrt{3}$	Cont.	0.2190	(2)
pdf → $f_X(x) = \begin{cases} \frac{x}{4}, & 1 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$	Cont.	$\int_1^2 \frac{x}{4} dx = \frac{x^2}{8}\Big\|_1^2 = \frac{3}{8} \approx 0.3750$	(2)
$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{27}, & 0 \le x < 1, \\ \frac{x^x}{27}, & 1 \le x \le 3, \\ 1, & x > 3. \end{cases}$	Cont.	$F_X(2) - F_X(1) = \frac{2^2}{27} - \frac{1^1}{27} = \frac{1}{9} \approx 0.1111$	(2*)
$X = 1 + \cos(\Theta)$ where $\Theta \sim \mathcal{U}(0, 2\pi)$			(2*)

For this question, you do not need to provide the reason or show your calculation. However, you may use the space below or the last page of the exam for your calculation.

$X \sim \mathcal{E}(3)$ ← λ

$$f_X(x) = \begin{cases} 3e^{-3x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$P[1 < X \le 2] = \int_1^2 3e^{-3x} dx = \cancel{3} \frac{e^{-3x}}{-\cancel{3}} \Big|_1^2 = -e^{-6} - (-e^{-3}) = e^{-3} - e^{-6} \approx 0.0473$$

$X \sim \mathcal{N}(1, 3)$ ← m, μ ; σ_X^2

$$P[1 < X \le 2] = F_X(2) - F_X(1) = \Phi\Big(\underbrace{\frac{2-1}{\sqrt{3}}}_{0.58}\Big) - \Phi\Big(\frac{\overbrace{1-1}^{0}}{\sqrt{3}}\Big)$$

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right) \quad = 0.7190 - 0.5 = 0.2190$$

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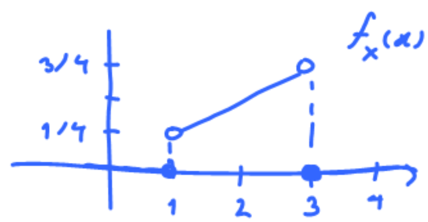
Problem 2. (6 pt) Let $X \sim$ Binomial$(36, 1/216)$. (For example, roll three dice 36 times and let X be the number of times a triple 6 appears.)

(a) (3 pt) Find $P[X > 0]$.

(b) (3 pt) Use Poisson approximation to find $P[X > 0]$.

Problem 3. (12 pt) A random variable X has probability density function

$$f_X(x) = \begin{cases} \frac{x}{4}, & 1 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

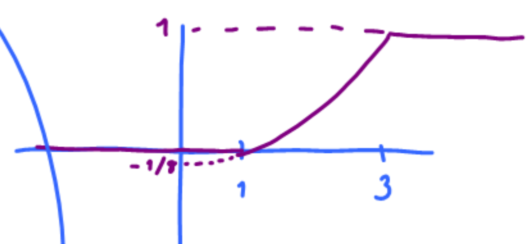


(a) (3 pt) Find and carefully sketch $F_X(x)$.

$$F_X(x) = \int_{-\infty}^{x} f_X(\mu)\,d\mu = \begin{cases} 0, & x \le 1 \\ & \\ 1, & x > 3 \end{cases}$$

For $1 < x < 3$: $F_X(x) = \int_1^x \frac{\mu}{4}\,d\mu = \left.\frac{\mu^2}{8}\right|_1^x = \frac{x^2}{8} - \frac{1}{8}$

$$F_X(x) = \begin{cases} 0, & x < 1 \\ (x^2-1)/8, & 1 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

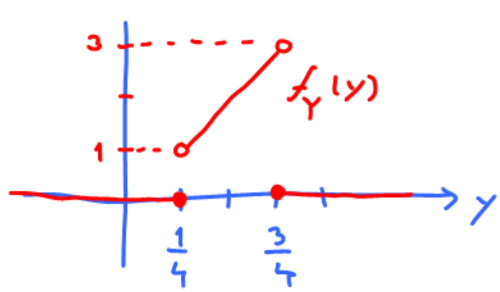


(b) (3 pt) Let $Y = \frac{X}{4}$. Find and carefully sketch the pdf of Y.

$Y = aX + b \qquad a = \frac{1}{4} \qquad b = 0$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = 4 f_X(4y) = 4 \times \begin{cases} 4y/4, & 1 < 4y < 3, \\ 0, & \text{otherwise} \end{cases}$$

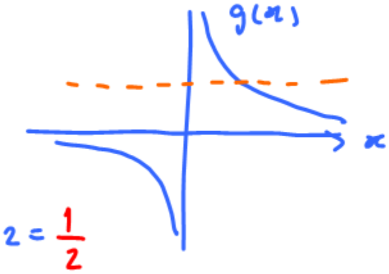
$$= \begin{cases} 4y, & \frac{1}{4} < y < \frac{3}{4}, \\ 0, & \text{otherwise.} \end{cases}$$



$P[V = \overset{5}{v}] = P[g(X) = \overset{5}{v}] = \sum_{x:\, g(x)=v} \underbrace{P[X=x]}_{0} = 0$ D [Nov 4]

countable ← there are at most one x that satisfies $v = g(x)$

(c) (5 pt) Let $V = \frac{1}{X}$. $= g(X)$ $g(x) = \frac{1}{x}$



(i) (2 pt) Find $\mathbb{E}V$.

$$\mathbb{E}V = \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_1^3 \frac{1}{\cancel{x}} \frac{\cancel{x}}{4}\, dx = \frac{1}{4} \times 2 = \frac{1}{2}$$

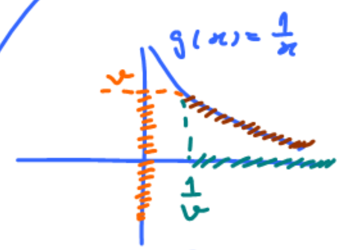
(ii) (1 pt*) Find the cdf of V.

Because $X \in (1,3)$, we know that $V = \frac{1}{X} \in \left(\frac{1}{3}, 1\right) \Rightarrow F_V(v) = \begin{cases} 0, & v < \frac{1}{3}, \\ \frac{1}{8}\left(9 - \frac{1}{v^2}\right), & \frac{1}{3} \le v \le 1, \\ 1, & v > 1 \end{cases}$

when $v \in \left[\frac{1}{3}, 1\right]$,

$$F_V(\overset{\frac{2}{3}}{v}) = P[V \le \overset{\frac{2}{3}}{v}] = P\left[\frac{1}{X} \le \overset{\frac{2}{3}}{v}\right] = P\left[X \ge \frac{1}{v}\right]$$

$$= \int_{1/v}^{\infty} f_X(x)\, dx = \int_{1/v}^{3} \frac{x}{4}\, dx = \frac{1}{8}\left(9 - \frac{1}{v^2}\right)$$



$-\frac{1}{v^2} = -v^{-2}$

(iii) (2 pt) Find the pdf of V.

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v \le \frac{1}{3}, \\ \frac{1}{\cancel{8}4}\left(\frac{\cancel{2}}{v^3}\right), & \frac{1}{3} < v < 1, \\ 0, & v \ge 1, \end{cases} = \begin{cases} \frac{1}{4v^3}, & \frac{1}{3} < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

V is a cont. RV

⇓

$f_V(v)$ @ $v = \frac{1}{3}, 1$ can be any real numbers ⇒ We set them to be 0.

(d) (1 pt***) Let $Z = X^X$. Find $f_Z(4)$.

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Problem 4. (23 pt) The random variable X has pmf

$$p_X(x) = \begin{cases} \frac{c}{x}, & x = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 pt) Check that the constant c must be $\frac{2}{3}$ and that $\mathbb{E}X = \frac{4}{3}$.

$\sum_x p_X(x) = 1 \Rightarrow \frac{c}{1} + \frac{c}{2} = 1$

$\frac{3}{2}c = 1$

$c = \frac{2}{3}$ ✓

$\mathbb{E}X = \sum_x x\, p_X(x)$

$= \cancel{1} \times \frac{c}{\cancel{1}} + \cancel{2} \times \frac{c}{\cancel{2}} = 2c = \frac{4}{3}$ ✓

(b) (11 pt) Let $V = \frac{1}{X}$.

(i) (3 pt) Find the pmf of V.

x	$p_X(x)$	$v = 1/x$
1	2/3	1/1 = 1
2	1/3	1/2

$$p_V(v) = \begin{cases} 2/3, & v = 1, \\ 1/3, & v = 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) (8 pt) Find the following quantities:

i. (3 pt) $\mathbb{E}[V]$

ii. (3 pt) $\mathbb{E}[V^2]$

iii. (2 pt) $\operatorname{Var} V$

(c) (10 pt) Now, consider another random variable Y. The pmf of Y is not known. However, suppose we know that

$$\mathbb{E}Y = 2, \quad \operatorname{Var} Y = 1, \quad \text{and } X \perp\!\!\!\perp Y.$$

(i) (1 pt*) Your friend, who seems to know the pmf of Y, calculate $P[|Y-2| \geq 2]$ and get $\frac{1}{3}$. Explain how this result is impossible.

(ii) (2 pt) Are X and Y uncorrelated?

(iii) (7 pt) Find the following quantities:

i. (2 pt) $\operatorname{Cov}[X, Y]$

ii. (1 pt) $\operatorname{Cov}[3X+5, 4Y]$

iii. (2 pt) $\mathbb{E}[XY]$

iv. (1 pt) $\mathbb{E}\left[\frac{Y}{X}\right]$

v. (1* pt) $\operatorname{Var}\left[\frac{Y}{X}\right]$

Problem 5. (21 pt) Random variables X and Y have the following joint pmf

$$p_{X,Y}(x,y) = \begin{cases} c(x+y), & x \in \{1,3\} \text{ and } y \in \{1,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (4 pt) Check that $c = \frac{1}{16}$ and then find the joint pmf matrix $P_{X,Y}$. Note that there are only four possible pairs (x,y) which have nonzero probabilities.

$\sum_{x,y} p_{X,Y}(x,y) = 1$

$2c + 4c + 4c + 6c = 1 \Rightarrow 16c = 1 \Rightarrow c = \frac{1}{16}$

$$P_{X,Y} \quad \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 1 & 2c & 4c \\ 3 & 4c & 6c \end{array} \quad \begin{array}{l} \\ \longrightarrow 6c \\ \longrightarrow 10c \end{array}$$

$\downarrow$ $6c = \frac{3}{8}$ $\quad$ $\downarrow$ $10c = \frac{5}{8}$

(b) (4 pt) Find the pmf $p_X(x)$ and the pmf $p_Y(y)$.

$$p_Y(y) = \begin{cases} 3/8, & y = 1, \\ 5/8, & y = 3, \\ 0, & \text{otherwise.} \end{cases} \qquad p_X(x) = \begin{cases} 3/8, & x = 1, \\ 5/8, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(c) (1 pt) Are X and Y identically distributed?

Yes

$p_X(c) = p_Y(c) \quad \forall c.$

(d) (3 pt) Find $\mathbb{E}X$ and $\mathbb{E}Y$.

$$\mathbb{E}X = \mathbb{E}Y = \sum_x x\, p_X(x) = 1 \times \frac{3}{8} + 3 \times \frac{5}{8} = \frac{18}{8} = \frac{9}{4}$$

(e) (3 pt) Find $\mathbb{E}[XY]$.

$$P_{X,Y} \quad \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 1 & 2c & 4c \\ 3 & 4c & 6c \end{array} \qquad xy \quad \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 1 & 1 & 3 \\ 3 & 3 & 9 \end{array}$$

$$\mathbb{E}[XY] = \sum_{(x,y)} xy\, p_{X,Y}(x,y)$$

$$= 1 \times 2c + 3 \times 4c + 3 \times 4c + 9 \times 6c$$

$$c = \frac{1}{16} \Rightarrow 5$$

(f) (2 pt) Are X and Y uncorrelated?

$\mathbb{E}[XY] \neq \mathbb{E}[X]\,\mathbb{E}[Y]$

$5 \qquad \frac{9}{4} \times \frac{9}{4}$

No.

(g) (2 pt) Are X and Y independent?

No, ("independent" implies "uncorrelated"

"Not uncorrelated" implies "Not independent")

$P_{X,Y}(1,1) = \frac{2}{16} = \frac{1}{8}$

$\neq$

$P_X(1) = \frac{3}{8} \times P_Y(1) = \frac{3}{8}$

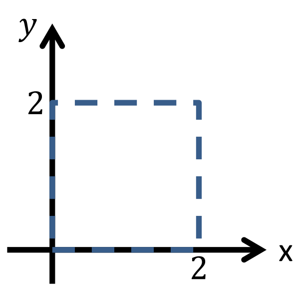
(h) (2 pt) Find Var$[X+Y]$.

$\mathrm{Var}[X+Y] = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\underbrace{\mathrm{Cov}[X,Y]}_{\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]}$

~~Problem~~ 6. (8 pt) Random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le y \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (4 pt) In the picture below, specify the region of nonzero pdf. Then, find the value of c.



(b) (4 pt) Find $\mathbb{E}X$.

Problem 7. (6 pt)

(a) (2 pt) Consider an exponential random variable $X \sim \mathcal{E}(3)$. Find the characteristic function $\varphi_X(v)$ of X.

(b) (4 pt) Suppose $Y = X_1 + X_2 + X_3 + X_4$ where X_1, X_2, X_3, and X_4 are i.i.d. $\mathcal{E}(3)$.

(i) (2 pt) Find the characteristic function $\varphi_Y(v)$ of Y.

(ii) (2 pt) Use $\varphi_Y(v)$ to find $\mathbb{E}Y$.

Problem 8. (6 pt) For the following parts, there will be no partial credit. Put only your final answers on this page. No explanation is required. You may use the space provided on the next page for you calculation.

(a) (1 pt + 1 pt*) Consider a function

$$g(x) = \begin{cases} 0, & 0 < x < \frac{4}{9}, \\ 1, & \frac{4}{9} \le x < \frac{8}{9}, \\ 2, & \text{otherwise.} \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(0,1)$. Which family of random variables does Y belong? What are its parameters?

(b) (2 pt*) Let $X \sim$ Binomial$(4800, 1/4)$. Recall that we can view a Binomial random variable as a sum of i.i.d. Bernoulli random variables. Therefore, we can closely approximate the probability $P[1188 < X \le 1212]$ by $2\Phi(z) - 1$ where Φ is the standard normal cdf.
Find z and the corresponding $2\Phi(z) - 1$.

(c) (1 pt**) Suppose X and Y are bivariate Gaussian random variables with

$$\mathbb{E}X = 2,\ \mathbb{E}Y = 3,\ \sigma_X = 2,\ \sigma_Y = 1,\ \rho_{X,Y} = \frac{1}{2}.$$

Find the values of the constants a and b which minimize

$$\mathbb{E}\left[(X - (aY + b))^2\right].$$

(d) (1 pt) Do not forget to submit your formula sheet with your exam.

Problem 9. Extra Credit (1 pt): What is the room number of Dr.Prapun's office? Hint: BKD3_ _ _ - _.

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