

EXAM 2 — Name

ID

*Lecturer: Prapun Suksompong, Ph.D.***Instructions**

- (a) Including this cover page, there are 12 pages. The last page is the formula sheet.
- (b) Read these instructions and the questions carefully.
- (c) Closed book. Closed notes.
- (d) Basic calculators, e.g. FX-991MS, are permitted, but borrowing is not allowed.
- (e) Allocate your time wisely. Some easy questions give many points.
- (f) The use of communication devices including mobile phones is prohibited in the examination room. Put all of your communication devices in your bag and leave it at the front of the examination room.
- (g) Do not forget to write your first name and the last three digits of your ID in the spaces provide on the top of each examination page, starting from page 2.
- (h) Write down all the steps that you have done to obtain your answers. You may not get any credit even when your final answer is correct without showing how you get your answer.
- (i) Some points are reserved for reducing answers into their simplest forms.
- (j) Do not cheat.
- (k) Do not panic.

Problem 1. (25pt) Random variables X and Y have the following joint pmf

$$p_{X,Y}(x,y) = \begin{cases} c(x+y), & x \in \{1, 3\} \text{ and } y \in \{2, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (1pt) Check that $c = 1/20$.

(b) (2pt) Find $P[X^2 + Y^2 = 13]$.

(c) (2pt) Find $p_X(x)$.

(d) (2pt) Find $\mathbb{E}X$.

(e) (2pt) Find $p_{Y|X}(y|1)$. Note that your answer should be of the form

$$p_{Y|X}(y|1) = \begin{cases} ?, & y = 2, \\ ?, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

(f) (2pt) Find $\mathbb{E}[Y|X = 1]$.

(g) (2pt) Find $p_{Y|X}(y|3)$.

(h) (2pt) Find $\mathbb{E}[Y|X = 3]$.

(i) (2pt) **Use** iterated expectation to find $\mathbb{E}Y$ from your answers in parts (c), (e) and (g).

(j) (2pt) Find $\mathbb{E}[XY]$.

(k) (2pt) Check that $\text{Cov}[X, Y] = -\frac{1}{25}$.

(l) Let $Z = X + Y$.

(i) (2pt) Find the pmf of Z .

(ii) (2pt) Find $\mathbb{E}Z$.

Problem 2. (14pt) Suppose X is uniformly distributed on the interval $(1, 2)$. ($X \sim \mathcal{U}(1, 2)$.) Let $Y = \frac{1}{X^2}$.

(a) (2pt) Plot the pdf $f_X(x)$ of X .

(b) (2pt) Plot the cdf $F_X(x)$ of X .

(c) (2pt) Find $P[\sin(X) > 0]$. Assume that X is in radians.

(d) (5pt) Find $f_Y(y)$.

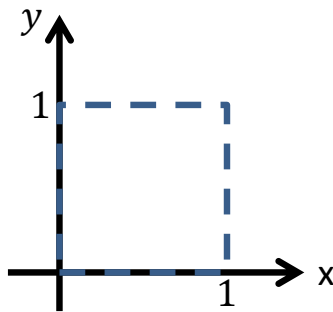
(e) (3pt) Find $\mathbb{E}Y$.

Problem 3. (16pt) Random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (1pt) Check that $c = 2$.

(b) (1pt) In the picture below, specify the region of nonzero pdf.



(c) (2pt) Find the marginal density $f_X(x)$.

(d) (2pt) Check that $\mathbb{E}X = \frac{2}{3}$.

(e) (2pt) Find the marginal density $f_Y(y)$.

(f) (2pt) Find $\mathbb{E}Y$

(g) (2pt) Find $\mathbb{E}[XY]$

(h) (2pt) Are X and Y uncorrelated?

(i) (2pt) Are X and Y independent?

Problem 4. (6pt) Let X_1 and X_2 be i.i.d. $\mathcal{E}(1)$ (exponential pdf with rate parameter $\lambda = 1$).

(a) (2pt) Find $P[X_1 = X_2]$.

(b) (1pt) Find $P[X_1^2 + X_2^2 = 13]$.

(c) (1pt*) Define $Y = \min\{X_1, X_2\}$. (For example, when $X_1 = 6$ and $X_2 = 4$, we have $Y = 4$.) Describe the random variable Y . Does it belong to any known family of random variables? If so, what is/are its parameters?

(d) (1pt*) Define $Y = \min\{X_1, X_2\}$ and $Z = \max\{X_1, X_2\}$. Find $f_{Y,Z}(2, 1)$.

(e) (1pt**) Define $Y = \min \{X_1, X_2\}$ and $Z = \max \{X_1, X_2\}$. Find $f_{Y,Z}(1, 2)$.

Problem 5. (8pt) Consider i.i.d. random variables X_1, X_2, \dots, X_{10} . Define the sample mean M by

$$M = \frac{1}{10} \sum_{k=1}^{10} X_k.$$

Let

$$V_1 = \frac{1}{10} \sum_{k=1}^{10} (X_k - \mathbb{E}[X_k])^2.$$

and

$$V_2 = \frac{1}{10} \sum_{k=1}^{10} (X_k - M)^2.$$

Suppose $\mathbb{E}[X_k] = 1$ and $\text{Var}[X_k] = 2$.

(a) (3pt*) Find the variance of the product: $\text{Var}[X_1 X_2]$.

(b) (1pt) Find $\mathbb{E}[M]$.

(c) (1pt) Find $\text{Var}[M]$.

(d) (2pt) Find $\mathbb{E}[V_1]$.

(e) (1pt**) Find $\mathbb{E}[V_2]$.

Problem 6. (9pt) Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-2, 2)$.

Remark: The function g operates like a **full-wave rectifier** in that if a positive input voltage X is applied, the output is $Y = X$, while if a negative input voltage X is applied, the output is $Y = -X$.

(a) (2pt) Find $\mathbb{E}Y$.

(b) (4pt) Plot the cdf of Y .

(c) (3pt) Find the pdf of Y .

Problem 7. (11 pt) Suppose a random variable X has density

$$f_X(x) = \frac{1}{4} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + c\delta(x - 1).$$

(a) (3 pt) Find c .

(b) (2 pt) Find $P[X = 1]$

(c) (2 pt) Find $P[X \leq 0]$

(d) (2 pt) Find $\mathbb{E}[X]$

(e) (2 pt) Find $\text{Var } X$

Problem 8. (7pt) Recall that the characteristic function $\varphi_X(v) = \mathbb{E}[e^{jvX}]$ can be used to find the moments $\mathbb{E}[X^k]$ of a random variable X . In particular,

$$\left. \frac{d^k}{dv^k} \varphi_X(v) \right|_{v=0} = j^k \mathbb{E}[X^k]. \quad (2.1)$$

In this problem, assume that $X \sim \mathcal{N}(m, \sigma^2)$. This means the characteristic function of X is given by

$$\varphi_X(v) = \mathbb{E}[e^{jvX}] = e^{jvm - \frac{1}{2}v^2\sigma^2}. \quad (2.2)$$

(a) (3pt) Use (2.1) and (2.2) to show that $\mathbb{E}X = m$.

(b) (3pt) Use (2.1) and (2.2) to show that $\mathbb{E}[X^2] = \sigma^2 + m^2$.

(c) (1pt*) Find $\mathbb{E}[X^3]$ when $m = 3$ and $\sigma = 2$.

Problem 9. (6pt) Kakashi and Gai are eternal rivals. Kakashi is a little stronger than Gai and hence for each time that they fight, the probability that Kakashi wins is 0.55. In a competition, they fight n times (where n is odd). We will assume that the results of the fights are independent. The one who wins more will win the competition.

(a) (3pt) Suppose $n = 3$, what is the probability that Kakashi wins the competition.

- (b) (3pt) The stronger person (Kakashi) should win the competition if n is very large. (By the law of large numbers, the proportion of fights that Kakashi wins should be close to 55%.) However, because the results are random and n can not be very large, we can not guarantee that Kakashi will win. However, it may be good enough if the probability that Kakashi wins the competition is greater than 0.85.

We want to find the minimal value of n such that the probability that Kakashi wins the competition is greater than 0.85.

Let N be the number of fights that Kakashi wins among the n fights. Then, we need

$$P \left[N > \frac{n}{2} \right] \geq 0.85. \quad (2.3)$$

Use the central limit theorem and Table 3.1 or Table 3.2 from [Yates and Goodman] to approximate the minimal value of n such that (2.3) is satisfied.