Name...ID...........................Section........1........Seat No............

Sirindhorn International Institute of Technology
Thammasat University

Midterm Examination: Semester 1 / 2016

Course Title: ECS315 (Probability and Random Processes)

Instructor: Asst. Prof. Dr.Prapun Suksompong

Date/Time: October 13, 2016 / 13:30 - 16:30

Instructions:

- This examination has ...10... pages (including this cover page).
- Conditions of Examination:
  ....Closed book
  (No dictionary, ☐ No calculator ☑ Calculator (e.g. FX-991) allowed)
  ....Open book
  ......✓...**Semi-Closed book** (.......1........sheet(s) ☑ 1 page ☐ both sides of A4 paper note)
  This sheet must be hand-written.
  Do not modify (,e.g., add/underline/highlight) content on the sheet inside the exam room.
  It should be **submitted with the exam**.
  Other requirements are specified on the course web site.
  (-10 pt if not following the requirements.)
- **Read these instructions and the questions carefully**.
- Students are not allowed to be out of the examination room during examination. Going to the restroom may result in score deduction.
- Turn off all communication devices and place them with other personal belongings in the area designated by the proctors or outside the test room.
- Write your name, student ID, and seat number clearly in the spaces provided on the top of this sheet. Then, write your **first name and the last three digits of your ID** in the spaces provided on the top of each page of your examination paper, starting from page 2.
- The back of each page will not be graded; it can be used for calculations of problems that do not require explanation.
- The examination paper is not allowed to be taken out of the examination room. Also, do not remove the staple. Violation may result in score deduction.
- Unless instructed otherwise**, *write down all the steps*** that you have done to obtain your answers.
  - When applying formula(s), state clearly which formula(s) you are applying before plugging-in numerical values.
  - You may not get any credit even when your final answer is correct without showing how you get your answer.
  - Formula(s) not discussed in class can be used. However, derivation must also be provided.
  - Exception: The 1-pt parts are graded solely on your answers. For these parts, there is no partial credit and it is not necessary to write down your explanation.
- When not explicitly stated/defined, all notations and definitions follow ones given in lecture.
- Some points are reserved for *accuracy* of the answers and also for reducing answers into their *simplest* forms.
- Points marked with * indicate challenging problems.
- Do not cheat. Do not panic. **Allocate your time wisely.**
- Don't forget to submit your fist online self-evaluation form by the end of today.

Problem 1. (11.5 + 0.5* pt) For each of the sets provided in the first column of the table below, indicate (by putting a Y(es) or an N(o) in the appropriate cells of the table) whether it is "finite", "infinite", "countably infinite", "uncountable". Explanation is not needed. Note that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of all natural numbers.

Sets	Finite	Infinite	Countably Infinite	Uncountable
$\{1, 2, 3, \ldots, 8\}$				
the power set of $\{1, 2, 3, \ldots, 8\}$				
the set of all real-valued x satisfying $\cos(x) = 0$				
the set of all real-valued x satisfying $\cos(x) \geq 0.9$				
$\mathbb{N}$				
the power set of $\mathbb{N}$				

Problem 2. (5 pt)

(a) (3 pt) Calculate the following quantities:

(i) (1 pt) 5!

(ii) (1 pt) $\binom{10}{2}$

(iii) (1 pt) $(10)_2$

(b) (2 pt) Without the help of your calculator, show how to simplify the following expression

binomial theorem with $x=1$, $y=1$, $n=1000$

$$\binom{1000}{0} + \binom{1000}{1} + \binom{1000}{2} + \binom{1000}{3} + \cdots + \binom{1000}{1000} = 2^{1000}$$

$$\binom{1000}{2} + \binom{1000}{3} + \cdots + \binom{1000}{1000} = 2^{1000} - 1 - 1000$$

Problem 3. (4 pt) Suppose we sample 3 objects from a collection of 5 distinct objects. Calculate the number of different possibilities when

(a) the sampling is ordered and performed with replacement

(b) the sampling is ordered and performed without replacement

(c) the sampling is unordered and performed without replacement

(d) the sampling is unordered and performed with replacement

Problem 4. ($2 \times 6 = 12$ pt)

(a) Calculate the number of different results when we permute

(i) ABCD

(ii) AAABBCC

(iii) 111||

(b) What is the coefficient of $x^{13}y^2$ in the expansion of $(x+y)^{15}$?

(c) Calculate the number of quadruples (x_1, x_2, x_3, x_4) satisfying

$$x_1 + x_2 + x_3 + x_4 = 100.$$

Assume all variables are nonnegative integers.

1111111111 ||| — 100 1s, 3 walls

$$\frac{103!}{100!\,3!} = \frac{103 \times 102 \times 101}{6} = \ldots$$

(d) A bin of 40 parts contains ten that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective.

Problem 5. (8 pt) Consider a random experiment whose sample space is $\{a, b, c, d, e\}$ with probabilities 0.2, 0.1, 0.1, 0.3, and 0.3, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Find the following probabilities. No explanation is needed.

$P(A) =$

$P(B) =$

$P(B^c) =$

$P(A \cap B) =$

$P(A \cup B) =$

$P(B|A) =$

$P(A|B) =$

$P(A|B^c) =$

Problem 6. (10 pt) Consider the following sequences of 1s and 0s which summarize the data obtained from 16 testees in a disease testing experiment.

```
 D: 0 1 1 0 0 0 0 1 1 1 1 0 1 0 1 1

TP: 0 0 0 0 1 0 1 1 0 0 1 1 0 0 1 1
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The results in the i-th column are for the i-th testee. The "`D`" row indicates whether each of the testees actually has the disease under investigation. The "`TP`" row indicates whether each of the testees is tested positive for the disease.

Numbers "1" and "0" correspond to "True" and "False", respectively.

Suppose we randomly pick a testee from this pool of 16 persons. Let D be the event that this selected person actually has the disease. Let T_P be the event that this selected person is tested positive for the disease.

Find the following probabilities. No explanation is needed here.

$P(D) =$ $\qquad$ $P(T_P) =$

$P(T_P \cap D) =$ $\qquad$ $P(T_P \cap D^c) =$

$P(T_P|D) =$ $\qquad$ $P(T_P|D^c) =$

$P(D|T_P) =$ $\qquad$ $P(D|T_P^c) =$

$P(T_P^c|D) =$ $\qquad$ $P(T_P \cup D) =$

Problem 7. (10 pt) Let A and B be events for which

$$P(A) = 1/3,\ P(B) = 3/4,\ \text{and } P(A \cup B) = 5/6.$$

(a) (2 pt) Find $P(A \cap B)$ $= P(A) + P(B) - P(A \cup B)$

$= \frac{1}{3} + \frac{3}{4} - \frac{5}{6} = \frac{3}{12} = \frac{1}{4}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) (2 pt) Are A and B independent? Don't forget to provide some explanation.

$P(A \cap B) \overset{?}{=} P(A)P(B)$ $\Rightarrow$ Yes.

$\frac{1}{4} \overset{\checkmark}{=} \frac{1}{3} \times \frac{3}{4}$

(c) (2 pt) Find $P(A|B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$

$P(A|B) \overset{?}{=} P(A) = \frac{1}{3}$

(d) (4 pt) Suppose the collection of three events E_1, E_2, and E_3 partitions the sample space. Furthermore, suppose $P(A|E_1) = \frac{1}{4}$ and $P(A|E_2) = \frac{1}{5}$.

(i) (3 pt) Suppose $P(E_2) = \frac{1}{3}$. **Use** "Form 1" of the Bayes' theorem to find $P(E_2|A)$.

$$P(E_2|A) = \frac{P(A|E_2)P(E_2)}{P(A)} = \frac{1/5 \times 1/3}{1/3} = \frac{1}{5}$$

(ii) (1* pt) Is it possible to have $P(A|E_3) = \frac{1}{6}$?

Problem 8. ($2 \times 6 = 12$ pt) [Digital Communication] A certain binary-symmetric channel has a crossover probability (bit-error rate) of 0.2. Assume bit errors occur independently.

(a) Suppose we input bit sequence "101101" into this channel.

(i) What is the probability that the output is "101101"?

(ii) What is the probability that the output is "100001"?

(iii) What is the probability that exactly 2 bits are in error at the channel output ?

(iv) What is the probability that there is at least one bit error at the channel output?

(b) Suppose we keep inputting bits into this channel. What is the probability that the *first* bit error at the output occurs on the fifth bit?

(c) To improve the system's reliability, a 3-bit repetition code is used. At the channel output, the decoder uses "majority rule". Find the new (info bit) error probability for this system.

Problem 9. (5 pt) Suppose that for the Land of Oz, 1 in 5 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 80% of the time. (The test is 80% accurate.)

(a) (1 pt) What is $P(-|\text{HIV})$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) (2 pt) Find the probability that a randomly chosen person tests positive.

(c) (2 pt) Find the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

Problem 10. (2 pt) Consider the outcome from a random experiment in which you roll a 10-sided fair dice. We define the following random variable from the outcomes of this experiment:

$$Y(\omega) = (\omega - 7)^2.$$

Find $P[Y > 10]$.

Problem 11. (9 pt) Consider a sample space $\Omega = \{1, 3, 4\}$. Suppose, for $\omega = 1, 3, 4$, we have

$$P(\{\omega\}) = c\omega$$

for some constant c.

(a) (2 pt) Check that $c = 1/8$.

(b) (4 pt) Define $A = \{1, 3\}$ and $B = \{1, 4\}$.

(i) Find $P(A)$.

(ii) Find $P(A \cap B)$.

(c) (3 pt) Define a random variable X by $X(\omega) = \frac{12}{\omega}$.

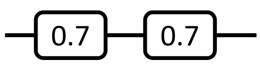
(i) (1 pt) What are the possible values of X?

(ii) (1 pt) Find $P[X = 3]$.

(iii) (1 pt) Find $P[X > 3]$.

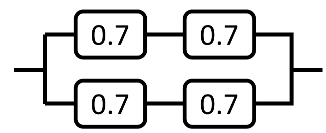
Problem 12. (6 pt) For each part below, what is the probability that the circuit operates? The circuit in each part operates if and only if there is a path of functional devices from left to right. The probability that each device functions is shown on the diagram. Assume that devices fail independently. Write your answer under the corresponding circuit. Explanation is not needed

(a) (3 pt)



(b) (2 pt)

(c) (1* pt)



Problem 13. (4 pt)

(a) (2 pt) Suppose $P(A_1 \cap A_2^c) = 1/4$ and $P(A_1^c \cap A_2) = 1/6$. Find *all* values of $P(A_1 \cap A_2)$ that make A_1 and A_2 independent.

(b) (2* pt) Suppose $P(B_1 \cap B_2^c \cap B_3^c) = 1/5$, $P(B_1^c \cap B_2 \cap B_3^c) = 2/15$, and $P(B_1^c \cap B_2^c \cap B_3) = 1/10$. Find *all* values of $P(B_1 \cap B_2 \cap B_3)$ that make the three events B_1, B_2, and B_3 independent. Explanation is not required for this part.

Problem 14. (1 pt) Make sure that you write your name and ID on every page. (Read the instruction on the cover page). Do not forget to submit your study sheet with your exam. Also, the online self-evaluation form is due by the end of today.