ECS 315: Probability and Random Processes **2013/1**

EXAM 1 — Name ID

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) Conditions of Examination:

- Closed book
- Calculator (e.g. FX-991MS) allowed)

(b) Read these instructions and the questions carefully.

(c) Students are not allowed to be out of the examination room during examination. Going to the restroom may result in score deduction.

(d) Turn off all communication devices and place them with other personal belongings in the area designated by the proctors or outside the test room.

(e) Write your name, student ID, section, and seat number clearly in the spaces provided on the top of this sheet. Then, write your first name and the last three digits of your ID in the spaces provided on the top of each page of your examination paper, starting from page 2.

(f) The examination paper is not allowed to be taken out of the examination room. Violation may result in score deduction.

(g) Unless instructed otherwise, write down all the steps that you have done to obtain your answers.

- You may not get any credit even when your final answer is correct without showing how you get your answer.
- Exception: The 1-pt questions will be graded on your answers. For these questions, because there is no partial credit, it is not necessary to write down your explanation.

(h) When not explicitly stated/defined, all notations and definitions follow ones given in lecture.

(i) Some points are reserved for accuracy of the answers and also for reducing answers into their simplest forms.

(j) Points marked with * indicate challenging problems.

(k) Do not cheat. Do not panic. Allocate your time wisely.

Problem 1 (M2013)**.** (12 pt) Harry rolls a fair **dice** *five* times. Let X_i be the result obtained from the ith roll.

(a) (2 pt) List all the outcomes $(X_1, X_2, X_3, X_4, X_5)$ such that

$$X_1 \times X_2 \times X_3 \times X_4 \times X_5 = 2.$$

There should be five of these.

(b) (3 pt) What is the probability that $X_1 \times X_2 \times X_3 \times X_4 \times X_5 = 2$?

(c) (3 pt) What is the probability that $X_1 \times X_2 \times X_3 \times X_4 \times X_5 = 15$?

(d) (1*+1 pt) What is the probability that $X_1 \times X_2 \times X_3 \times X_4 \times X_5 = 1125$?

(e) (1* pt) What is the probability that $X_1 < X_2 < X_3 < X_4 < X_5$?

(f) (1* pt) What is the probability that $X_1 \leq X_2 \leq X_3 \leq X_4 \leq X_5$?

Problem 2 (M2013)**.** (27 pt) No explanation is needed for this problem. Simply indicate your answers.

(a) (8 pt) Let A and B be events for which

$$P(A) = P(B) = 0.4 \text{ and } P(A \cup B) = 0.7.$$

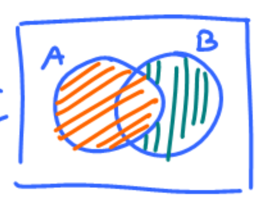
(i) (2 pt) Find $P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.4 + 0.4 - 0.7 = 0.1$

(ii) (2 pt) Find $P(A \cap (A^c \cap B))$ $= P(\emptyset)$

$= 0$



(iii) (2 pt) Find $P(A \cup (A^c \cap B))$ $= P(A \cup B) = 0.7$

(iv) (2 pt) Are A and B independent? No

$P(A \cap B) = 0.1 \neq P(A) \times P(B) = 0.4 \times 0.4$

(a.i.)

(b) (10 pt) Let C and D be events for which

$$P(C|D) = P(D) = 0.2.$$

$= \frac{P(C \cap D)}{P(D)}$

(i) (2 pt) Find $P(C \cap D)$ $= P(C|D)P(D) = 0.2 \times 0.2 = 0.04$

(ii) (2 pt) Find $P(C^c \cap D)$ $= P(C^c|D)\underbrace{P(D)}^{0.2} = (1-0.2)(0.2) = 0.8 \times 0.2$

$P(C^c|D) = 1 - P(C|D)$ $= 0.16$

(iii) (2* pt) Find the range of possible values for $P(C)$.

(iv) (4 pt) Suppose we also know that C and D are independent.

i. (2 pt) Find $P(C)$

$C \perp\!\!\!\perp D$

$P(C) = P(C|D) = 0.2$

ii. (2 pt) Find $P(C|D^c)$

$C \perp\!\!\!\perp D \Rightarrow C \perp\!\!\!\perp D^c$

$P(C|D^c) = P(C) = 0.2$

(c) (9 pt) Suppose events E, F, and G are independent. Let

$$P(E) = P(F) = P(G) = \frac{1}{3}.$$

(i) (1 pt) Are events E, F, and G pairwise independent?

(ii) (2 pt) Find $P(E|F \cap G)$. (Note that this is the probability that event E happens, knowing that the event $F \cap G$ has already occurred.)

(iii) (2 pt) Find $P(E \cap F \cap G|G)$. (Note that this is the probability that event $E \cap F \cap G$ happens, knowing that the event G has already occurred.)

(iv) (2 pt) Find $P(E \cup F \cup G|G)$.

(v) (2* pt) Find $P(E|F \cup G)$.

Problem 3 (M2013)**.** (10 pt) A random experiment has 24 **equiprobable** outcomes:

$$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x\}$$

Let A denote the event $\{a, b, c, d, e, f, g, h, i, j, k, l\}$, and let B denote the event $\{i, j, k, l, m, n, o, p\}$.

(a) Determine the following:

(i) (3 pt) $P(A)$

(ii) (3 pt) $P(A \cup B^c)$

(iii) (3 pt) Are A and B independent?

(b) (1* pt) Construct a nontrivial[1] event C so that A, B, and C are independent. Do not forget to show that your answer satisfies ALL conditions for testing three independent events.

[1]Here, nontrivial event means that it is not empty nor the whole sample space.

Problem 4 (M2013)**.** (13 pt)

(a) (8 pt) Suppose that for the Land of Oz, 1 in 10 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 90% of the time. (The test is 90% accurate.) We would like to find the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

(i) (2 pt) What is $P(-|HIV)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(ii) (3 pt) ***Use*** the law of total probability to find $P(+)$, the probability that a randomly chosen person tests positive.

(iii) (3 pt) ***Use*** Bayes' formula to find $P(HIV|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

(b) (5 pt) You have three coins in your pocket. The first one is a fair coin. The second one is biased with probability of heads $\frac{1}{3}$. The third one is biased with probability of heads $\frac{1}{4}$. All of them look identical. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins?

Problem 5 (M2013)**.** (10 pt) Kakashi and Gai are eternal rivals. For each time that they fight, the probability that Kakashi wins is 0.5. In this problem, we will consider the results of their fights. Assume that the results of the fights are independent.

(a) (8 pt) Consider the results from the **first five** of their fights.

(i) (4 pt) What is the probability that Kakashi wins exactly three times?

(ii) (4 pt) What is the probability that Kakashi wins at least once?

(b) (1* pt) What is the probability that Kakashi's first win is on their 6th fight?

(c) (1* pt) What is the probability that Kakashi's first win happens before or on their 6th fight?

Problem 6 (M2013)**.** (18 pt) Consider a sample space $\Omega = \{1, 2, 3\}$. Suppose, for $\omega = 1, 2, 3$, we have

$$P(\{\omega\}) = \frac{c - |\omega - 2|}{4}$$

for some constant c.

(a) (3 pt) Check that $c = 2$.

(b) (12 pt) Define a random variable X by $X(\omega) = (\omega - 4)^2$.

(i) (3 pt) Find $P[X = 9]$.

(ii) (3 pt) Find and plot the pmf of X.

(iii) (3 pt) Find $P[X \leq 5]$.

(iv) (3 pt) Plot the cdf of X.

(c) (3 pt) Define another random variable Y by $Y(\omega) = \sqrt{|(\omega - 2|}$.

(i) (1 pt) Is Y a uniform random variable?

(ii) (1 pt) Is Y a Bernoulli random variable?

(iii) (1 pt) Is Y a Binomial random variable?

Problem 7 (M2013)**.** (8 pt) The cdf of a random variable X is plotted in Figure 1.1.

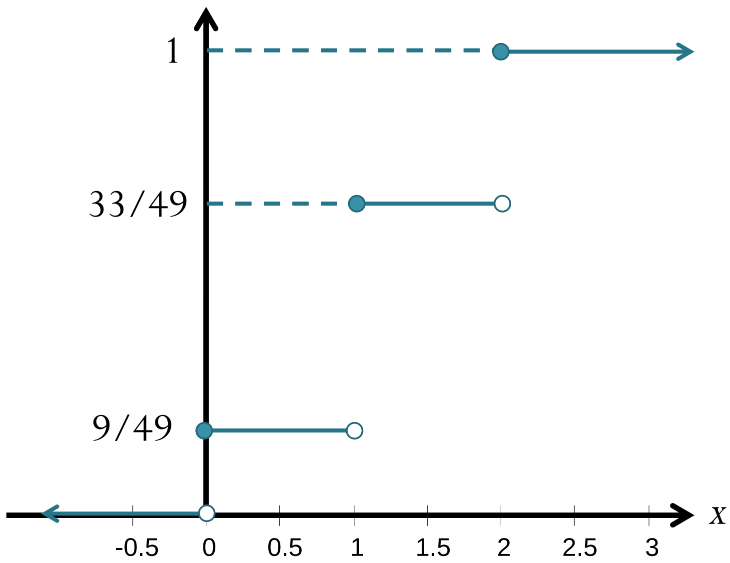


Figure 1.1: CDF of X for Problem 7

(a) (4 pt) Find and carefully plot the pmf $p_X(x)$.

(b) (2 pt) Find $P[X > 1]$.

(c) (1* pt) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, or Poisson)

(d) (1* pt) Suppose the random variable X is generated (in MATLAB) for a large number of times. Predict the average of the generated results.

Problem 8 (M2013)**.** Extra Credits

(a) Who was Dr.Prapun's advisor when he was at Cornell?

(b) Fill in the blanks:

"Les ________________ les plus importantes de la vie ne sont en effet, pour la plupart, que des problèmes de ________________."