ECS 315: Probability and Random Processes **2018/1**

HW Solution 5 — Due: Sep 25, 4 PM

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Problem 1. Series Circuit: The circuit in Figure 5.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

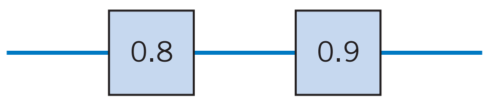

Figure 5.1: Circuit for Problem 1

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Problem 2. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.

Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 3. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 4. You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By Bayes' theorem, the probability that one of the fair coins has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + p \times \frac{1}{3}} = \boxed{\frac{1}{1 + p}}.$$

Alternative Solution: Let F_1, F_2, U and H be the events that "the selected coin is the first fair coin", "the selected coin is the second fair coin", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the events $F1, F2$, and U are equally likely:

$$P(F_1) = P(F_2) = P(U) = \frac{1}{3}.$$

For fair coins, the conditional probability of heads is $\frac{1}{2}$ and for the unfair coin, the conditional probability of heads is p:

$$P(H|F_1) = P(H|F_2) = \frac{1}{2}, \quad P(H|U) = p.$$

The probability that one of the fair coins has been selected given that it lands heads up is $P(F_1 \cup F_2|H)$. Now F_1 and F_2 are disjoint events. Therefore,

$$P(F_1 \cup F_2|H) = P(F_1|H) + P(F_2|H).$$

By Bayes' theorem,

$$P(F_1|H) = \frac{P(H|F_1)P(F_1)}{P(H)} \quad \text{and} \quad P(F_2|H) = \frac{P(H|F_2)P(F_2)}{P(H)}.$$

Therefore,

$$P(F_1 \cup F_2|H) = \frac{P(H|F_1)P(F_1)}{P(H)} + \frac{P(H|F_2)P(F_2)}{P(H)} = \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H)}.$$

Note that the collection of three events F_1, F_2, and U partitions the sample space. Therefore, by the total probability theorem,

$$P(H) = P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U).$$

Plugging the above expression of $P(H)$ into our expression for $P(F_1 \cup F_2|H)$ gives

$$\begin{aligned} P(F_1 \cup F_2|H) &= \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U)} \\ &= \frac{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3} + p\times\frac{1}{3}} = \boxed{\frac{1}{1+p}}. \end{aligned}$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. In his book *Chances: Risk and Odds in Everyday Life*, James Burke says that there is a 72% chance a polygraph test (lie detector test) will catch a person who is, in fact, lying. Furthermore, there is approximately a 7% chance that the polygraph will falsely accuse someone of lying. [Brase and Brase, 2011, Q4.2.26]

(a) If the polygraph indicated that 30% of the questions were answered with lies, what would you estimate for the actual percentage of lies in the answers?

(b) If the polygraph indicated that 70% of the questions were answered with lies, what would you estimate for the actual percentage of lies?

Solution: Let AT and AL be the events that "the person actually answers the truth" and "the person actually answers with lie", respectively. Also, let PT and PL be the events that "the polygraph indicates that the answer is the truth" and "the polygraph indicates that the answer is a lie", respectively.

We know, from the provided information, that $P(PL|AL) = 0.72$ and that $P(PL|AT) = 0.07$.

Applying the total probability theorem, we have

$$\begin{aligned} P(PL) &= P(PL|AL)P(AL) + P(PL|AT)P(AT) \\ &= P(PL|AL)P(AL) + P(PL|AT)(1 - P(AL)). \end{aligned}$$

Solving for $P(AL)$, we have

$$P(AL) = \frac{P(PL) - P(PL|AT)}{P(PL|AL) - P(PL|AT)} = \frac{P(PL) - 0.07}{0.72 - 0.07} = \frac{P(PL) - 0.07}{0.65}.$$

(a) Plugging in $P(PL) = 0.3$, we have $P(AL) = \boxed{0.3538}$.

(b) Plugging in $P(PL) = 0.7$, we have $P(AL) = \boxed{0.9692}$.

Problem 6. Software to detect fraud in consumer phone cards tracks the number of metropolitan areas where calls originate each day. It is found that 1% of the legitimate users originate calls from two or more metropolitan areas in a single day. However, 30% of fraudulent users originate calls from two or more metropolitan areas in a single day. The proportion of fraudulent users is 0.01%. If the same user originates calls from two or more metropolitan areas in a single day, what is the probability that the user is fraudulent? [Montgomery and Runger, 2010, Q2-144]

Solution: Let F denote the event of fraudulent user and let M denote the event of originating calls from multiple (two or more) metropolitan areas in a day. Then,

$$\begin{aligned} P(F|M) &= \frac{P(M|F)P(F)}{P(M|F)P(F) + P(M|F^c)P(F^c)} = \frac{1}{1 + \frac{P(M|F^c)}{P(M|F)} \times \frac{P(F^c)}{P(F)}} \\ &= \frac{1}{1 + \frac{\frac{1}{100}}{\frac{30}{100}} \times \frac{\frac{9999}{10^4}}{\frac{1}{10^4}}} = \frac{1}{1 + \frac{9999}{30}} = \frac{30}{30 + 9999} = \frac{30}{10029} \approx \boxed{0.0030}. \end{aligned}$$