In general, from Section 6.1 we know that

$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$

6.2 Event-based Independence

Plenty of random things happen in the world all the time, most of which have nothing to do with one another. If you toss a coin and I roll a dice, the probability that you get heads is 1/2 regardless of the outcome of my dice. Events that are unrelated to each other in this way are called *independent*.

Definition 6.33. Two events A, B are called (statistically[27]) ***independent*** if

$\perp$ disjoint

$\perp\!\!\!\perp$ independent

Ш ECS332's Shah func.

$$P(A \cap B) = P(A)\, P(B) \tag{9}$$

- Notation: $A \perp\!\!\!\perp B$
- Read "A and B are independent" or "A is independent of B"
- We call (9) the ***multiplication rule*** for probabilities.
- If two events are not independent, they are ***dependent***. Intuitively, if two events are dependent, the probability of one changes with the knowledge of whether the other has occurred.

6.34. Intuition: Again, here is how you should think about independent events: "If one event has occurred, the probability of the other does not change."

$\frac{P(A \cap B)}{P(B)} = P(A)$ $\qquad$ $\frac{P(A \cap B)}{P(A)} = P(B)$

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B). \tag{10}$$

In other words, "the unconditional and the conditional probabilities are the same". We can almost use (10) as the definitions for independence. This is what we mentioned in 6.8. However, we use (9) instead because it (1) also works with events whose probabilities are zero and (2) also has clear symmetry in the expression (so that $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp A$ can clearly be seen as the same). In fact, in 6.37, we show how (10) can be used to define independence with extra condition that deals with the case when zero probability is involved.

[27]Sometimes our definition for independence above does not agree with the everyday-language use of the word "independence". Hence, many authors use the term "statistically independence" to distinguish it from other definitions.

Example 6.35. [25, Ex. 5.4] Which of the following pairs of events are independent?

(a) The card is a club, and the card is black.
 C $\qquad$ B

Method #1

$\underbrace{P(C\cap B)}_{\frac{13}{52}} \neq \underbrace{P(C)}_{\frac{13}{52}}\underbrace{P(B)}_{\frac{26}{52}}$

$\frac{13}{52} \neq \frac{13}{52}\ \frac{26}{52}$

$\frac{1}{4} \neq \frac{1}{4}\times\frac{1}{2}$

$\Rightarrow$ C and B are dependent.

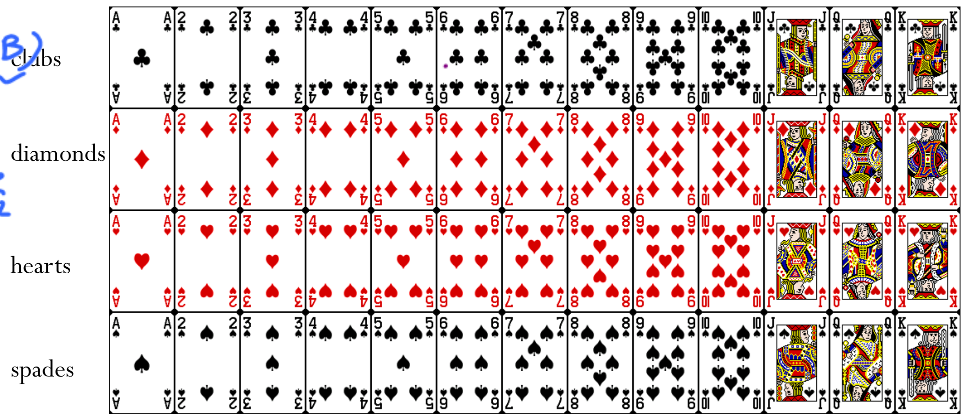


Figure 12: A Deck of Cards

Method #2

$\underbrace{P(B|C)}_{1} \neq \underbrace{P(B)}_{\frac{1}{2}}$

$1 \neq \frac{1}{2}$

$\Rightarrow$ C and B are dependent

(b) The card is a king, and the card is black.
 K $\qquad$ B

$P(K) = \frac{4}{52} = \frac{1}{13}$ $\qquad$ $\frac{1}{2}$

$\frac{1}{26} = \frac{2}{52} = P(K\cap B) \overset{\checkmark}{=} P(K)P(B)$ $\qquad$ $\Rightarrow$ K and B are independent

6.36. An event with probability 0 or 1 is independent of any event (including itself).

- In particular, $\emptyset$ and Ω are independent of any events.
- One can also show that an event A is independent of itself if and only if $P(A)$ is 0 or 1.

6.37. Now that we have 6.36, we can now extend the "practival definition" from 6.34 to include events with zero probabilities:

Two events A, B *with positive probabilities* are independent if and only if $P(B|A) = P(B)$, which is equivalent to $P(A|B) = P(A)$.

When A and/or B has zero probability, A and B are automatically independent.

6.38. When A and B have nonzero probabilities, the following statements are equivalent:

1) $A \perp\!\!\!\perp B$

2) $P(A \cap B) = P(A)P(B)$

3) $P(A|B) = P(A)$

4) $P(B|A) = P(B)$

complete proof in the extra part of the HW

6.39. The following four statements are equivalent:

$$A \perp\!\!\!\perp B, \quad A \perp\!\!\!\perp B^c, \quad A^c \perp\!\!\!\perp B, \quad A^c \perp\!\!\!\perp B^c.$$

$A \perp\!\!\!\perp B \Rightarrow P(A \cap B) = P(A)P(B)$

$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)P(B^c)$

Example 6.40. If $P(A|B) = 0.4$, $P(B) = 0.8$, and $P(A) = 0.5$, are the events A and B independent? [15]

Method 1

$P(A \cap B) \neq P(A)P(B)$

$P(B)P(A|B) = 0.8 \times 0.4 \neq 0.5 \times 0.8$

$\Rightarrow$ A and B are not independent

Method 2

$P(A|B) \neq P(A)$

$0.4 \neq 0.5$

$\Rightarrow$ A and B are not independent.

6.41. Keep in mind that **independent and disjoint** are ***not synonyms***. In some contexts these words can have similar meanings, but this is not the case in probability.

"Being disjoint" suggests "being dependent"

- If two events cannot occur at the same time (they are disjoint), are they independent? At first you might think so. After all, they have nothing to do with each other, right? Wrong! They have a lot to do with each other. If one has occurred, we know for certain that the other cannot occur. [17, p 12]

- To check whether A and B are disjoint, we only need to look at the sets themselves and see whether they have shared outcome(s). This can be answered without knowing probabilities.

  To check whether A and B are independent, we need to compute the probabilities $P(A)$, $P(B)$, and $P(A \cap B)$.

"Being disjoint" is a set property.

"Being independent" is a probabilistic property.

- Addition vs. multiplication:

  (a) If events A and B are disjoint, we calculate the probability of their union $A \cup B$ by adding the probabilities of A and B.

  If $A \perp B$, then $P(A \cup B) = P(A) + P(B) \cancel{- P(A \cap B)}^{0}$

  (b) For independent events A and B, we calculate the probability of their intersection $A \cap B$ by multiplying the probabilities of A and B.

  If $A \perp\!\!\!\perp B$, then $P(A \cap B) = P(A)P(B)$

- The two statements $A \perp B$ and $A \perp\!\!\!\perp B$ can occur simultaneously only when $P(A) = 0$ and/or $P(B) = 0$.

  $A \cap B = \emptyset$

  $P(A \cap B) = P(A)P(B)$

  $0 = P(\emptyset) = P(A)P(B) \Rightarrow P(A) = 0$ or $P(B) = 0$

  - Reverse is not true in general.

Example 6.42. Experiment of flipping a fair coin twice. $\Omega = \{HH, HT, TH, TT\}$. Define event A to be the event that the first flip gives a H; that is $A = \{HH, HT\}$. Event B is the event that the second flip gives a H; that is $B = \{HH, TH\}$. Note that even though the events A and B are not disjoint, they are independent.

$P(A) = \frac{2}{4} = \frac{1}{2}$

$P(B) = \frac{2}{4} = \frac{1}{2}$

$A \cap B = \{HH\} \neq \emptyset \Rightarrow A$ and B are not disjoint

$P(A \cap B) = \frac{1}{4} = P(A)P(B) \Rightarrow A$ and B are independent.

Example 6.43 (Slides)**.** ***Prosecutor's fallacy***: In 1999, a British jury convicted Sally Clark of murdering two of her children who had died suddenly at the ages of 11 and 8 weeks, respectively. A pediatrician called in as an expert witness claimed that the chance of having two cases of sudden infant death syndrome (SIDS), or "cot deaths," in the same family was 1 in 73 million. There was no physical or other evidence of murder, nor was there a motive. Most likely, the jury was so impressed with the seemingly astronomical odds against the incidents that they convicted. But where did the number come from? Data suggested that a baby born into a family similar to the Clarks faced a 1 in 8,500 chance of dying a cot death. Two cot deaths in the same family, it was argued, therefore had a probability of $(1/8,500)^2$ which is roughly equal to 1/73,000.000.

Did you spot the error? The computation assumes that successive cot deaths in the same family are *independent* events. This assumption is clearly questionable, and even a person without any medical expertise might suspect that genetic factors play a role. Indeed, it has been estimated that if there is one cot death, the next child faces a much larger risk, perhaps around 1/100. To find the probability of having two cot deaths in the same family, we should thus use conditional probabilities and arrive at the computation $1/8,500 \times 1/100$, which equals l/850,000. Now, this is still a small number and might not have made the jurors judge differently. But what does the probability 1/850,000 have to do with Sallys guilt? Nothing! When her first child died, it was certified to have been from natural causes and there was no suspicion of foul play. The probability that it would happen again without foul play was 1/100, and if that number had been presented to the jury, Sally would not have had to spend three years in jail before the verdict was finally overturned and the expert witness (certainly no expert in probability) found guilty of "serious professional misconduct."

You may still ask the question what the probability 1/100 has to do with Sallys guilt. Is this the probability that she is innocent? Not at all. That would mean that 99% of all mothers who experience two cot deaths are murderers! The number 1/100 is simply the probability of a second cot death, which only means that among all families who experience one cot death, about 1% will suffer through another. If probability arguments are used in court cases, it is very important that all involved parties understand some basic probability. In Sallys case, nobody did.

References: [14, 118–119] and [17, 22–23].

Definition 6.44. Three events A_1, A_2, A_3 are independent if and only if

$$\left.\begin{aligned} &(1) & P(A_1 \cap A_2) &= P(A_1)\, P(A_2) & & A_1 \perp\!\!\!\perp A_2 \\ &(2) & P(A_1 \cap A_3) &= P(A_1)\, P(A_3) & & A_1 \perp\!\!\!\perp A_3 \\ &(3) & P(A_2 \cap A_3) &= P(A_2)\, P(A_3) & & A_2 \perp\!\!\!\perp A_3 \end{aligned}\right\} \Rightarrow \text{pairwise independent}$$

$$(4) \quad P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$$

Remarks:

(a) When the first three equations hold, we say that the three events are ***pairwise independent***.

(b) We may use the term "mutually independence" to further emphasize that we have "independence" instead of "pairwise independence".

Definition 6.45. The events $A_1, A_2, \ldots, A_n$ are ***independent*** if and only if for any subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$,

$$P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = P\left(A_{i_1}\right) \times P\left(A_{i_2}\right) \times \cdots \times P\left(A_{i_k}\right).$$

- Note that part of the requirement is that

$$P\left(A_1 \cap A_2 \cap \cdots \cap A_n\right) = P\left(A_1\right) \times P\left(A_2\right) \times \cdots \times P\left(A_n\right).$$

  Therefore, if someone tells us that the events $A_1, A_2, \ldots, A_n$ are independent, then one of the properties that we can conclude is that

$$P\left(A_1 \cap A_2 \cap \cdots \cap A_n\right) = P\left(A_1\right) \times P\left(A_2\right) \times \cdots \times P\left(A_n\right).$$

- Equivalently, this is the same as the requirement that

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P\left(A_j\right) \quad \forall J \subset [n] \text{ and } |J| \geq 2$$

- Note that the case when $j = 1$ automatically holds. The case when $j = 0$ can be regarded as the $\emptyset$ event case, which is also trivially true.

6.46. Four events A, B, C, D are pairwise independent if and only if they satisfy the following six conditions:

$$\left.\begin{aligned}
P(A \cap B) &= P(A)P(B),\\
P(A \cap C) &= P(A)P(C),\\
P(A \cap D) &= P(A)P(D),\\
P(B \cap C) &= P(B)P(C),\\
P(B \cap D) &= P(B)P(D), \text{ and}\\
P(C \cap D) &= P(C)P(D).
\end{aligned}\right\} \quad \binom{4}{2} = 6 \text{ conditions}$$

They are independent if and only if they are pairwise independent (satisfy the six conditions above) and also satisfy the following five more conditions:

To check independence among five events, we need to check $\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

$$\left.\begin{aligned}
P(B \cap C \cap D) &= P(B)P(C)P(D),\\
P(A \cap C \cap D) &= P(A)P(C)P(D),\\
P(A \cap B \cap D) &= P(A)P(B)P(D),\\
P(A \cap B \cap C) &= P(A)P(B)P(C), \text{ and}
\end{aligned}\right\} \quad \binom{4}{3} = 4 \text{ conditions}$$

$$P(A \cap B \cap C \cap D) = P(A)P(B)P(C)P(D). \quad \binom{4}{4} = 1 \text{ condition}$$

$$(x+y)^5 = \sum_{k=0}^{5} \binom{5}{k} x^k y^{5-k}$$

$$2^5 = \binom{5}{0} + \binom{5}{1} + \underbrace{\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}}_{32-1-5 = 26}$$

In total, to check independence among four events, we need to check 11 conditions.

6 events $\Rightarrow 2^6 - \binom{6}{0} - \binom{6}{1} = 64 - 1 - 6 = 57$

Example 6.47. Suppose five events A, B, C, D, E are independent with

$$P(A) = P(B) = P(C) = P(D) = P(E) = \frac{1}{3}.$$

(a) Can they be (mutually) disjoint?

No, the sum of their probabilities already exceeds 1.

(b) Find $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{P(A)P(B)}$

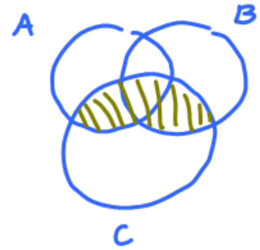
$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \times \frac{1}{3} = \frac{3+3-1}{9} = \frac{5}{9}$$

(c) Find $P((A \cup B) \cap C)$

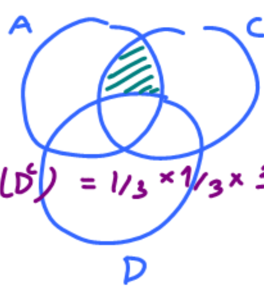
$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$

$= \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 = \frac{5}{27}$



(d) Find $P(A \cap C \cap D^c)$



$= P(A \cap C) - P(A \cap C \cap D)$

$= P(A)P(C) - P(A)P(C)P(D)$

$= P(A)P(C)(1 - P(D)) = P(A)P(C)P(D^c) = 1/3 \times 1/3 \times \frac{2}{3} = \frac{2}{27}$

$= (1/3)^2 - (1/3)^3 = 2/27$

(e) Find $P(A \cap B | C)$

$$= \frac{P((A \cap B) \cap C)}{P(C)} \overset{\perp\!\!\!\perp}{=} \frac{P(A)P(B)\cancel{P(C)}}{\cancel{P(C)}} = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

$= P(A \cap B)$

Review:

The following statements are equivalent

(1) Events A and B are independent
(2) $A \perp\!\!\!\perp B$
(3) $P(A\cap B) = P(A)P(B)$ ← good for checking independence
(4) $A \perp\!\!\!\perp B^c$
(5) $P(A\cap B^c) = P(A)P(B^c)$
(6) $A^c \perp\!\!\!\perp B$
(7) $P(A^c\cap B) = P(A^c)P(B)$
(8) $A^c \perp\!\!\!\perp B^c$
(9) $P(A^c\cap B^c) = P(A^c)P(B^c)$

Furthermore, if $P(A) > 0$ and $P(B) > 0$,

(10) $P(A|B) = P(A)$
(11) $P(B|A) = P(B)$

Independence for three events