

ECS 303 - Part 3C

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116

10. AC POWER ANALYSIS

10.2. Average Power

The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure.

DEFINITION 10.2.1. The average power is the average of the instantaneous power over one period.

Thus, the average power is given by $\cos(\alpha) = \cos(-\alpha)$

$$P = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i).$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and the current. Note that $p(t)$ is time varying while P does not depend on time.

10.2.2. Using the phasor forms of $v(t)$ and $i(t)$, which are $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, we obtain

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} \text{Re}\{\mathbf{V}\mathbf{I}^*\}.$$

$$\begin{aligned}\vec{V} &= V_m e^{j\theta_v} \\ \vec{I} &= I_m e^{j\theta_i} \\ \vec{I}^* &= I_m e^{-j\theta_i} \\ \vec{V}\vec{I}^* &= V_m I_m e^{j(\theta_v - \theta_i)}\end{aligned}$$

10.2.3. Two special cases:

Case 1: When $\theta_v = \theta_i$, the voltage and the current are in phase. This implies a **purely resistive circuit** or resistive load R , and

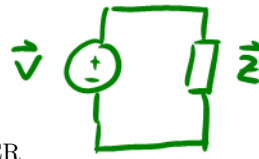
$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R,$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. This shows that a **purely resistive circuit (e.g. resistive load (R)) absorbs power all times.**

Case 2: When $\theta_v - \theta_i = \pm 90^\circ$, we have a **purely reactive circuit**, and

$$P = \frac{1}{2} V_m I_m \cos(90^\circ) = 0$$

showing that a **purely reactive circuit (e.g. a reactive load L or C) absorbs no average power.**



10.2.4. From Ohm's law, we have two more formula.

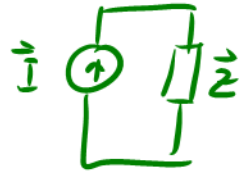
- (a) The average power absorbed by an impedance Z when a voltage V is applied across it is

$$P = \frac{1}{2} \text{Re}\{V\mathbf{I}^*\} = \frac{1}{2} \text{Re}\left\{V \frac{V^*}{Z^*}\right\} = \frac{1}{2} |V|^2 \text{Re}\left\{\frac{1}{Z^*}\right\} = \frac{1}{2} |V|^2 \frac{\text{Re}\{Z\}}{|Z|^2}$$

$$\text{Re}\left\{\frac{1}{Z^*}\right\} = \text{Re}\left\{\frac{Z}{|Z|^2}\right\}$$

- (b) The average power absorbed by an impedance Z when a current I flows through it is

$$P = \frac{1}{2} \text{Re}\{V\mathbf{I}^*\} = \frac{1}{2} \text{Re}\{I\mathbf{Z}\mathbf{I}^*\} = \frac{1}{2} |I|^2 \text{Re}\{Z\}$$



EXAMPLE 10.2.5. Calculate the average power absorbed by an impedance $Z = 30 - j70$ when a voltage $V = 120\angle 0^\circ$ is applied across it.

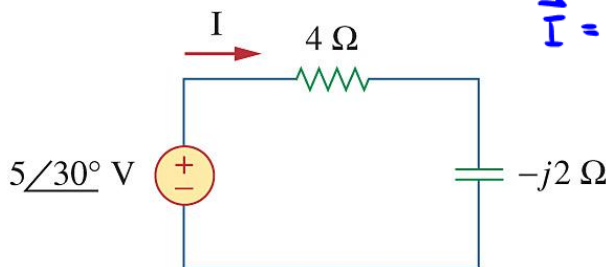
$$P = \frac{1}{2} \text{Re}\{V\mathbf{I}^*\} = \frac{1}{2} |V|^2 \frac{\text{Re}\{Z\}}{|Z|^2} = \frac{1}{2} \times 120^2 \times \frac{30}{30^2 + 70^2} = 37.24 \text{ W}$$

EXAMPLE 10.2.6. A current $I = 10\angle 30^\circ$ flows through an impedance $Z = 20\angle -22^\circ$. Find the average power delivered to the impedance.

$$P = \frac{1}{2} |I|^2 \text{Re}\{Z\} = \frac{1}{2} \times 10^2 \times 20 \times \cos(-22^\circ) = 1000 \times \cos(-22^\circ) = 927 \text{ W}$$

$$\text{Re}\{\vec{x}\} = |x| \cos(\theta_x)$$

EXAMPLE 10.2.7. For the circuit shown below, find the average power supplied by the source and the average power absorbed by the resistor.



$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - 2j}$$

$$P_{\text{source}} = -\frac{1}{2} |V|^2 \frac{\text{Re}\{Z\}}{|Z|^2}$$

$$= -\frac{1}{2} \times 5^2 \times \frac{4}{4^2 + 2^2} = -\frac{1}{2} \times \frac{25 \times 4}{20} = -2.5 \text{ W}$$

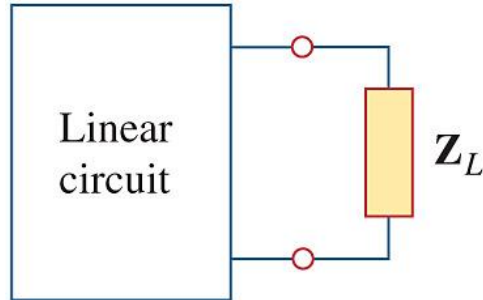
$$P_{4\Omega} = 2.5 \text{ W}$$

$$P_{4\Omega} = \frac{1}{2} \text{Re}\{\vec{V}_{4\Omega} \times \vec{I}_{4\Omega}^*\}$$

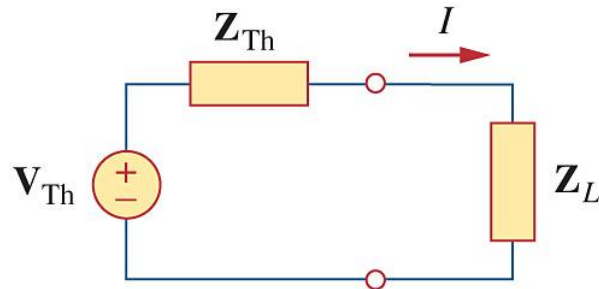
$$= \frac{1}{2} \text{Re}\left\{\vec{I}_{4\Omega} \times 4 \times \vec{I}_{4\Omega}^*\right\} = \frac{1}{2} |\vec{I}_{4\Omega}|^2 \times 4 = \frac{1}{2} \times 4 \times \frac{5^2}{4^2 + 2^2}$$

10.3. Maximum Average Power Transfer

Previously, we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load R_L . Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{Th}$. We now extend that result to ac circuits.



Consider an ac circuit which is connected to a load Z_L and is represented by its Thevenin equivalent.



In a rectangular form, the Thevenin impedance Z_{Th} and load impedance Z_L are

$$Z_{Th} = R_{Th} + jX_{Th}$$

$$Z_L = R_L + jX_L$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{Z_L + Z_{Th}},$$

and the average power delivered to the load is

$$P = \frac{1}{2} |\mathbf{I}|^2 R_L = \frac{1}{2} |\mathbf{V}_{Th}|^2 \frac{R_L}{(R_L + R_{Th})^2 + (X_L + X_{Th})^2}$$

Our objective is to adjust the load parameter R_L and X_L so that P is maximum. To do this we set $\frac{\partial P}{\partial R_L}$ and $\frac{\partial P}{\partial X_L}$ equal to zero.

Setting $\frac{\partial P}{\partial X_L} = 0$ gives

$$X_L = -X_{Th}.$$

Setting $\frac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2}$$

Hence, to get the maximum **average power** transfer, the load impedance \mathbf{Z}_L must be selected so that

$$X_L = -X_{Th} \text{ and } R_L = R_{Th},$$

i.e.,

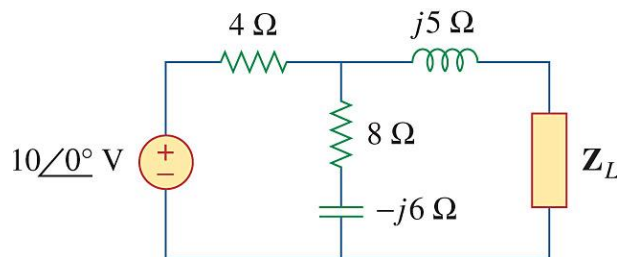
$$\mathbf{Z}_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^*$$

That is, for the maximum average power transfer, the load impedance \mathbf{Z}_L must be equal to the complex conjugate of the Thevenin impedance \mathbf{Z}_{Th} .

When $\mathbf{Z}_L = \mathbf{Z}_{Th}^*$, the maximum average power is

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}}.$$

EXAMPLE 10.3.1. Determine the load impedance \mathbf{Z}_L that maximizes the average power drawn from the circuit below. What is the maximum average power?



10.3.2. In a situation in which the load must be purely real; that is X_L must be 0. Then,

$$P = \frac{1}{2} |\mathbf{I}|^2 R_L = \frac{1}{2} |\mathbf{V}_{Th}^2| \frac{R_L}{(R_L + R_{Th})^2 + (X_{Th})^2}$$

Setting $\frac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2} = |\mathbf{Z}_{Th}|.$$

Hence, for maximum average power transfer to a purely resistive load, the load impedance is equal to the magnitude of the Thevenin impedance \mathbf{Z}_{Th} . In which case, the maximum average power is

$$P = \frac{1}{4} |\mathbf{V}_{Th}^2| \frac{1}{|\mathbf{Z}_{Th}| + R_{Th}}$$

Note that

$$|\mathbf{Z}_{Th}| + R_{Th} \geq R_{Th} + R_{Th} = 2R_{Th}$$

Hence,

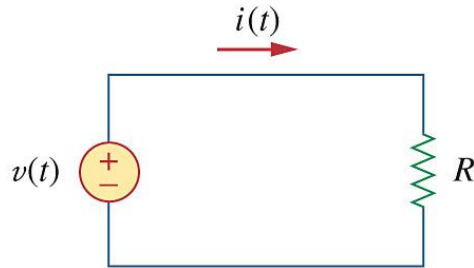
$$\frac{1}{4} |\mathbf{V}_{Th}^2| \frac{1}{|\mathbf{Z}_{Th}| + R_{Th}} \leq \frac{1}{8} \frac{|\mathbf{V}_{Th}^2|}{R_{Th}}.$$

10.4. Effective or RMS Value

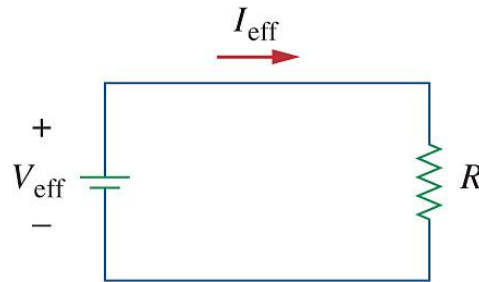
The idea of **effective value** arises from the need to measure the effectiveness of an ac voltage or current source in delivering power to a resistive load.

DEFINITION 10.4.1. The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

Consider the following ac and dc circuits,



our objective is to find the current I_{eff} that will transfer the same power to the resistor R as the sinusoid current i



The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T} \int_0^T i^2(t) R dt = \frac{R}{T} \int_0^T i^2(t) dt$$

while the power absorbed by the resistor in the dc circuit is

$$P = I_{\text{eff}}^2 R.$$

Equating the expressions and solving for I_{eff} , we obtain

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt}$$

Similarly, the effective value of the voltage is found in the same way as current; that is,

$$V_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T v^2(t) dt}$$

This indicates that the effective value is the square root of the mean (or average) of the square of the periodic signal. Thus, the effective value is often known as the **root mean square, or rms value** for short. We write

$$I_{\text{eff}} = I_{\text{rms}}, \quad V_{\text{eff}} = V_{\text{rms}}$$

Note:

- (a) The rms value of a constant is the constant itself.
- (b) For a sinusoid $i(t) = I_m \cos(\omega t + \theta_i)$, the effective value or rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2(\omega t + \theta_i) dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos(\omega t + \theta_v)$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

and the average power can be written in terms of the rms values as

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i).$$

Similarly, the average power absorbed by a resistor R can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R}.$$

10.5. Apparent Power and Power Factor

DEFINITION 10.5.1. The **apparent power** S (in VA) is the product of the rms values of voltage and current.

$$S = V_{rms}I_{rms}$$

Hence, the average power $P = S \cos(\theta_v - \theta_i)$.

DEFINITION 10.5.2. The **power factor** (pf) is the ratio of the average power to the apparent power.

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i).$$

Hence,

$$\text{average power } P = \text{apparent power } S \times \text{power factor pf.}$$

The angle $\theta_v - \theta_i$ is called the **power factor angle** which is equal to the angle of the load impedance if $\mathbf{V} = V_m \angle \theta_v$ is the voltage across the load and $\mathbf{I} = I_m \angle \theta_i$ is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle (\theta_v - \theta_i).$$

Alternately, define $\mathbf{V}_{rms} = \frac{\mathbf{V}}{\sqrt{2}} = V_{rms} \angle \theta_v$ and $\mathbf{I}_{rms} = \frac{\mathbf{I}}{\sqrt{2}} = I_{rms} \angle \theta_i$. The impedance can then be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle (\theta_v - \theta_i).$$

The power factor is the cosine of the phase difference between the voltage and current. It is also the cosine of the angle of the load impedance.

10.5.3. The value of the power factor pf ranges between 0 and 1.

- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and $\text{pf} = 1$. This implies that the average power is equal to the apparent power .
- For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$. Hence, $\text{pf} = 0$. In this case the average power is zero.
- In between these two extreme cases, pf is said to be **leading** or **lagging**. **Leading power factor** means that current leads voltage which implies a capacitive load. **Lagging power factor** means that current lags voltage, implying an inductive load.

10.6. Complex Power

The **complex power** (in VA) \mathbf{S} is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. \mathbf{S} is a complex quantity whose real part is the real or average power P and imaginary part is the **reactive** power Q .

Complex power: \mathbf{S}

$$\mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = V_{\text{rms}}I_{\text{rms}}\angle(\theta_v - \theta_i) = I_{\text{rms}}^2\mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}.$$

Note that all previously studied quantities can be derived from the complex power. That is,

The apparent power S is the magnitude of the complex power \mathbf{S} , i.e.,

$$S = |\mathbf{S}| = V_{\text{rms}}I_{\text{rms}} = \sqrt{P^2 + Q^2}.$$

The real or average power P is

$$P = \text{Re}\{\mathbf{S}\} = S \cos(\theta_v - \theta_i) = I_{\text{rms}}^2 R.$$

The reactive power Q is

$$Q = \text{Im}\{\mathbf{S}\} = S \sin(\theta_v - \theta_i) = I_{\text{rms}}^2 X.$$

The power factor pf is

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i) = \cos(\text{phase of } \mathbf{S}).$$

Three Phase Circuits

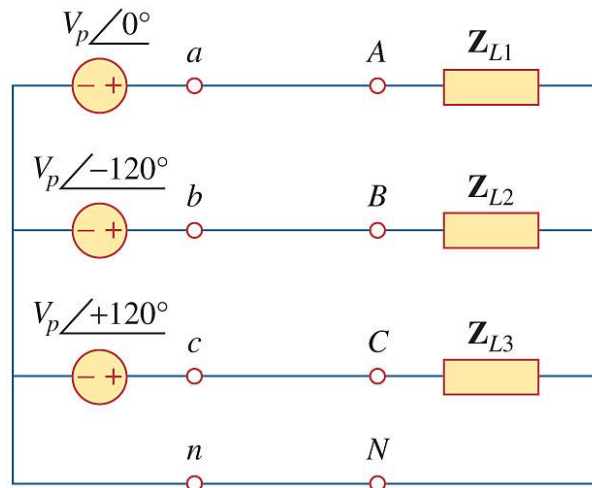
Up to this point, our study on AC circuit analysis has mainly focused on the systems referred to as *single-phase circuits*.



11.1. Polyphase Circuits

DEFINITION 11.1.1. Circuits or systems in which the **ac sources** operate at the **same frequency** but **different phases** are known as *polyphase*.

11.1.2. Three-phase four-wire system:

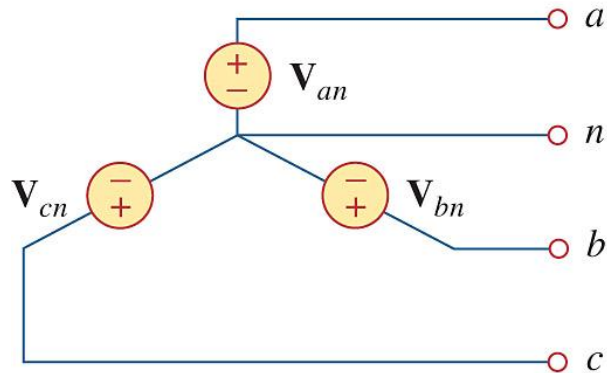


11.1.3. Nearly all electric power is generated and distributed in three-phase. Three-phase systems are useful because

- (a) the instantaneous power in a three-phase system can be constant (not pulsating), and
- (b) for the same amount of power, the three-phase system is more economical than the single-phase.

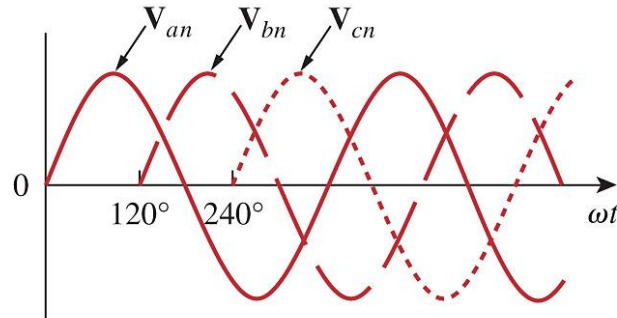
11.1.4. When the United States was in the “battle of the currents”, Tesla invented the first polyphase ac power system which greatly influenced the settlement of the ac versus dc debate in favor of ac.

11.1.5. A **three-phase system** contains a set of three voltage sources having the same frequency but out-of-phase.

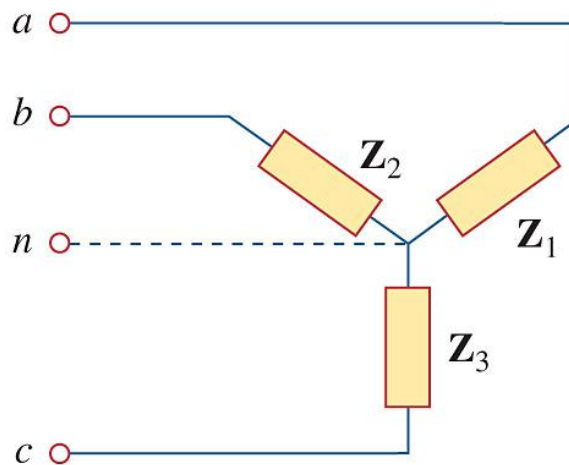


- These voltages are called **phase voltages**.
- a , b , and c are the name of each of the three phases, while n is the common node of the three phases. This node is known as **neutral**.
- The notation of each phase voltage then represents the voltage between each phase and the neutral. For example, V_{an} is the voltage between phase a and the neutral n .

- If the voltage sources have the same amplitude and frequency ω and are out of phase with each other by 120° , the voltages are said to be **balanced**.
 - This set of voltage sources is referred to as a *balanced three-phase voltage source*.



- The load is said to be unbalanced if the phase impedances are not equal in magnitude or phase.



- A **balanced three-phase circuit** is then defined as a circuit having a set of balanced three-phase voltage source connected with a balanced three-phase load.
 - In which case, the notations used at the load are in capital letters, i.e., A , B , C , and N , while those at the sources are in small letters as a , b , c and n .

DEFINITION 11.1.6. The phase sequence is the time order in which the voltages pass through their respective maximum values.

11.2. Balanced Three-Phase Voltages sources

11.2.1. For balanced three-phase voltages sources, suppose that V_p is the effective or rms value of the phase voltages. Because the three-phase voltages are 120° out of phase with each other, there are two possible combinations:

(a) The three phase voltages is

$$v_{an}(t) = V_p \cos(\omega t + \theta),$$

$$v_{bn}(t) = V_p \cos(\omega t + \theta - 120^\circ),$$

$$v_{cn}(t) = V_p \cos(\omega t + \theta - 240^\circ) = V_p \cos(\omega t + \theta + 120^\circ).$$

for some θ .

- This is known as the **abc sequence** or **positive sequence**.

- In this phase sequence, V_{an} leads V_{bn} , which in turn leads V_{cn} .

(b) The three phase voltages is

$$v_{an}(t) = V_p \cos(\omega t + \theta),$$

$$v_{cn}(t) = V_p \cos(\omega t + \theta - 120^\circ),$$

$$v_{bn}(t) = V_p \cos(\omega t + \theta - 240^\circ) = V_p \cos(\omega t + \theta + 120^\circ).$$

- This is known as the **acb sequence** or **negative sequence**.

- In this phase sequence, V_{an} leads V_{cn} , which in turn leads V_{bn} .

11.2.2. Important property for phase voltages:

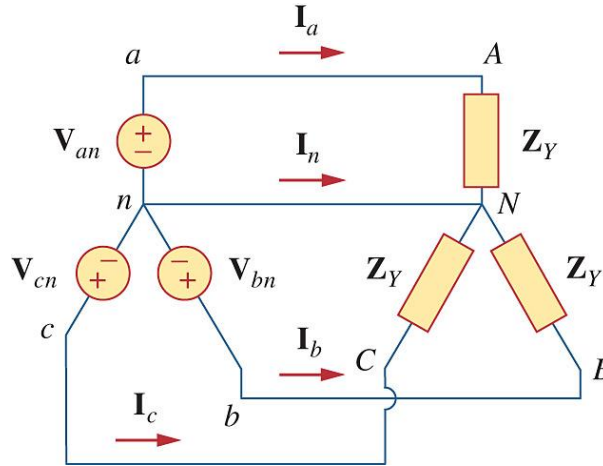
$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0.$$

and

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| = V_p.$$

11.3. Balanced Wye-Wye Connection

DEFINITION 11.3.1. A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balance Y-connected load.



- Although the impedance \mathbf{Z}_Y is the total load impedance per phase, it may also be regarded as the sum of the source impedance \mathbf{Z}_S , line impedance \mathbf{Z}_l and load impedance \mathbf{Z}_L for each phase.

Without loss of generality, we will assume positive sequence with $\theta = 0$. In which case, the **phase voltages** (or line-to-neutral voltages) are

$$\begin{aligned}\mathbf{V}_{an} &= V_p \angle 0^\circ, \\ \mathbf{V}_{bn} &= V_p \angle -120^\circ = \mathbf{V}_{an} \angle -120^\circ, \\ \mathbf{V}_{cn} &= V_p \angle +120^\circ = \mathbf{V}_{an} \angle +120^\circ.\end{aligned}$$

11.3.2. From the phase voltages, we can calculate the line-to-line voltages or simply **line voltages** \mathbf{V}_{ab} , \mathbf{V}_{bc} , and \mathbf{V}_{ca} (using KVL) as

$$\begin{aligned}\mathbf{V}_{ab} &= \mathbf{V}_{an} - \mathbf{V}_{bn} = \sqrt{3}V_p \angle 30^\circ \\ \mathbf{V}_{bc} &= \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p \angle -90^\circ \\ \mathbf{V}_{ca} &= \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p \angle -210^\circ\end{aligned}$$

Observe that

- the magnitude of the line voltages V_L is $\sqrt{3}$ times the magnitude of the phase voltages V_p ,
- the line voltages lead their corresponding phase voltages by 30° , and
- Notice that \mathbf{V}_{ab} leads \mathbf{V}_{bc} by 120° , and \mathbf{V}_{bc} leads \mathbf{V}_{ca} by 120° , so that the line voltages sum up to zero as do the phase voltages.

DEFINITION 11.3.3. The line current is the current in each line. The phase current is the current in each phase of the source or load.

- In the Y-Y system, the line current is the same as the phase current.
- It is natural and conventional to assume that line currents flow from the source to the load.

11.3.4. By Ohm's law, the line currents (which are the same as the phase currents) are given by

$$\begin{aligned}\mathbf{I}_a &= \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{V_p \angle 0^\circ}{\mathbf{Z}_Y}, \\ \mathbf{I}_b &= \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{V_p \angle -120^\circ}{\mathbf{Z}_Y} = \mathbf{I}_a \angle -120^\circ, \\ \mathbf{I}_c &= \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{V_p \angle +120^\circ}{\mathbf{Z}_Y} = \mathbf{I}_a \angle +120^\circ.\end{aligned}$$

Note that the three currents, \mathbf{I}_a , \mathbf{I}_b and \mathbf{I}_c have the same magnitude and out-of-phase with each other by $\angle 120^\circ$.

11.3.5. Because

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0,$$

we have

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} + \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} + \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{1}{\mathbf{Z}_Y} (\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn}) = 0$$

Alternatively, this fact is easily found from (11.3.4).

11.3.6. By KCL, the neutral line current is

$$\begin{aligned}\mathbf{I}_n &= -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c), \\ &= 0\end{aligned}$$

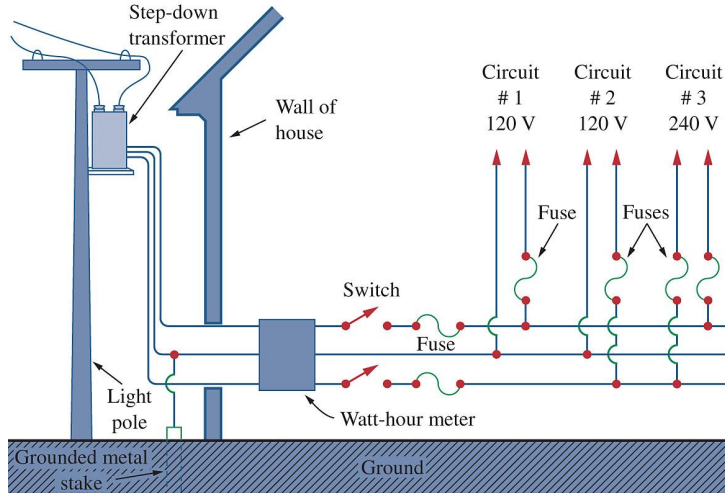
and the voltage across the neutral wire is

$$\mathbf{V}_{nN} = \mathbf{Z}_n \mathbf{I}_n = 0.$$

- The neutral line can be removed without affecting the system.

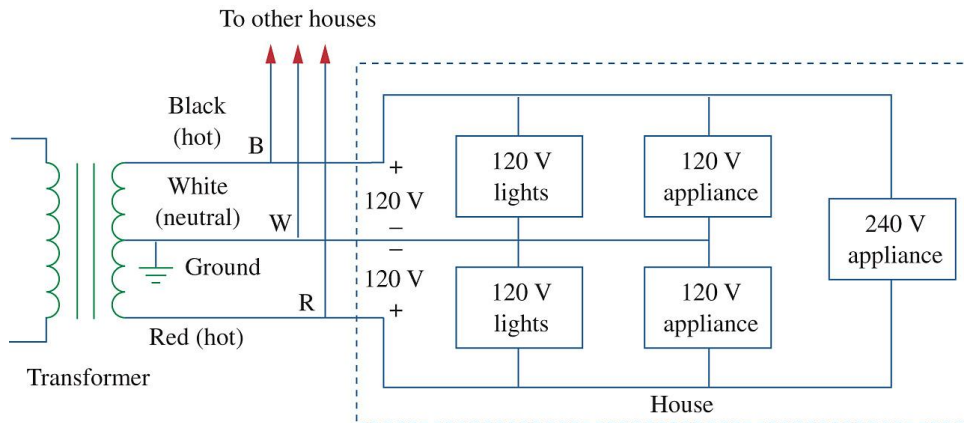
11.4. Residential Wiring in the United States

In the United States, most household lighting and appliances operate on 120-V, 60-Hz, single-phase alternating current. The local power company supplies the house with a three-wire ac system. Typically, as in Fig. 12.37, the line voltage of, say, 12,000 V is stepped down to 120/240 V with a transformer.



11.4.1. The three wires coming from the transformer are typically colored red (hot), black (hot), and white (neutral).

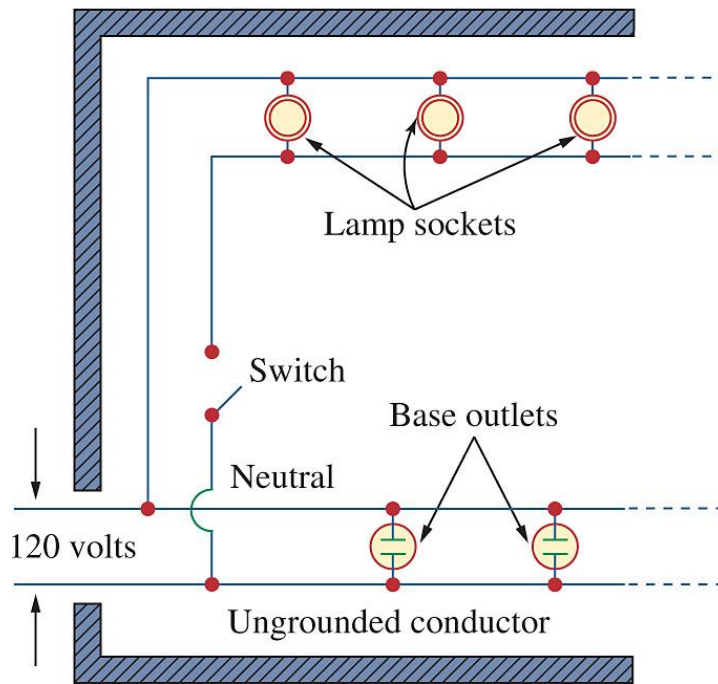
$$\mathbf{V}_W = 0\angle 0^\circ, \quad \mathbf{V}_B = 120\angle 0^\circ, \quad \mathbf{V}_R = 120\angle 180^\circ = -\mathbf{V}_B.$$



11.4.2. Note that

$$\mathbf{V}_{BR} = \mathbf{V}_B - \mathbf{V}_R = 2\mathbf{V}_B = 240\angle 0^\circ.$$

11.4.3. Since most appliances are designed to operate with 120 V, the lighting and appliances are connected to the 120-V lines. Also, all appliances are connected in parallel.



Heavy appliances that consume large currents, such as air conditioners, dishwashers, ovens, and laundry machines, are connected to the 240-V power line.