ECS 303 - Part 3B
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CHAPTER 9

Sinusoidal Steady State Analysis

9.1. General Approach

In the previous chapter, we have learned that the steady state response of a circuit to sinusoidal inputs can be obtained by using phasors. In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, superposition, and source transformations are applied in analyzing ac circuits.

Steps to analyze ac circuits, using phasor domain:

Step 1. Transform the circuit to the phasor or frequency domain.
- Not necessary if the problem is specified in the frequency domain.

Step 2. Solve the problem using circuit techniques (e.g., nodal analysis, mesh analysis, Thevenin's theorem, superposition, or source transformations)
- The analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Step 3. Transform the resulting phasor back to the time domain.

9.1.1. ac circuits are linear (they are just composed of sources and impedances)

9.1.2. The **superposition theorem** applies to ac circuits the same way it applies to dc circuits.

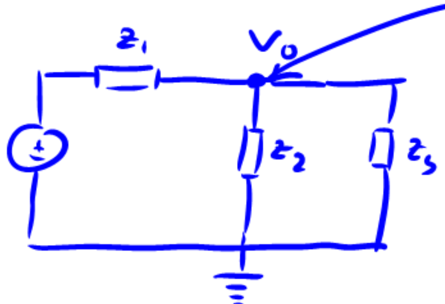
Example:

$V_s = 19.3 - 5.18j\ V_s$

$Z_1 = 60$

$Z_2 = -25j$

$Z_3 = 20j$



$$\frac{V_0 - V_s}{Z_1} + \frac{V_0}{Z_2} + \frac{V_0}{Z_3} = 0$$

$$V_0\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) = \frac{V_s}{Z_1}$$

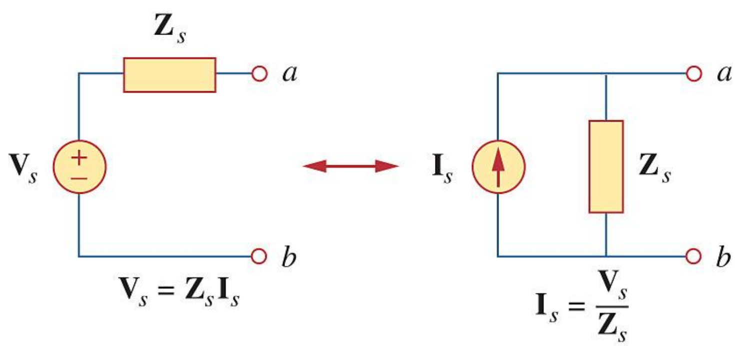
$$V_0 = \frac{V_s / Z_1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}$$

$$= 16.48 + 4.72j$$

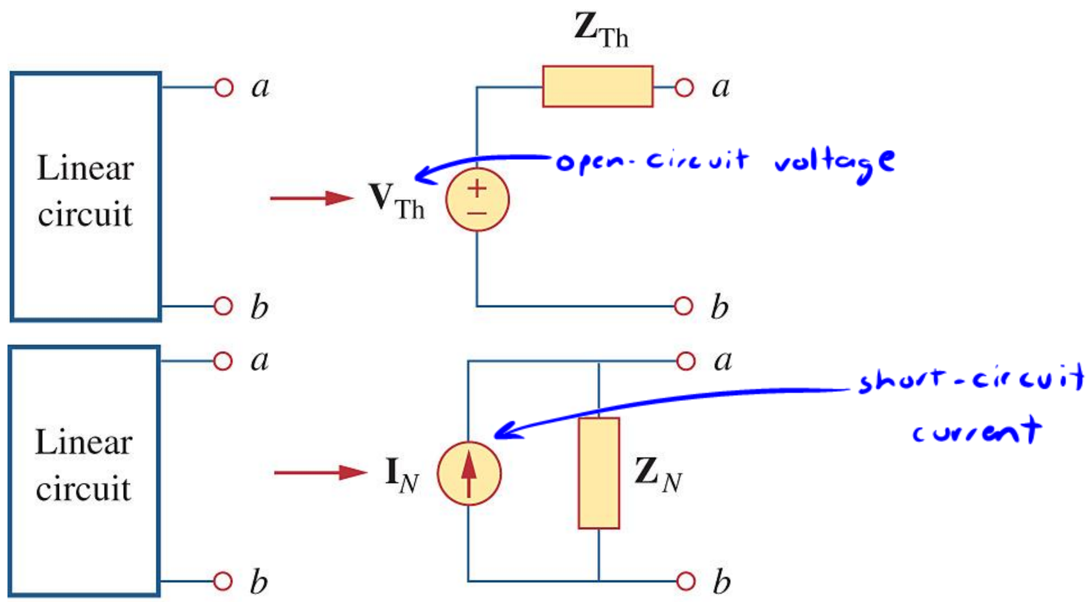
$$= 17.15 \angle 15.96^0$$

9.1.3. **Source transformation**:

$$\mathbf{V_s} = \mathbf{Z_s I_s}, \quad \mathbf{I_s} = \frac{\mathbf{V_s}}{\mathbf{Z_s}}.$$

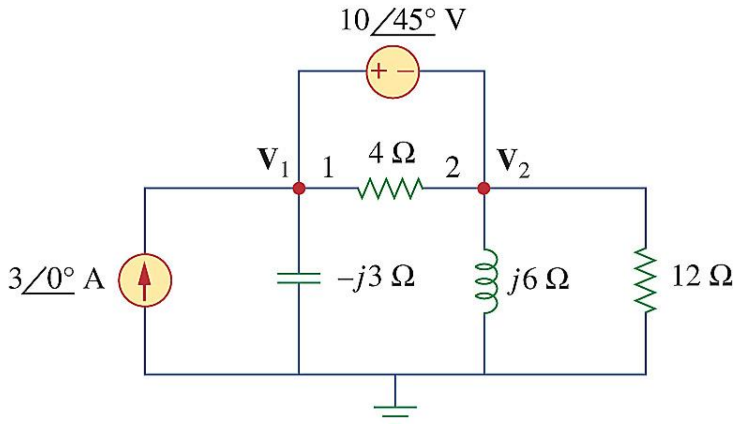


9.1.4. **Thevenin and Norton Equivalent circuits**:



$$\mathbf{V_{Th}} = \mathbf{Z_N I_N}, \quad \mathbf{Z_{Th}} = \mathbf{Z_N}$$

EXAMPLE 9.1.5. Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit below using nodal analysis.

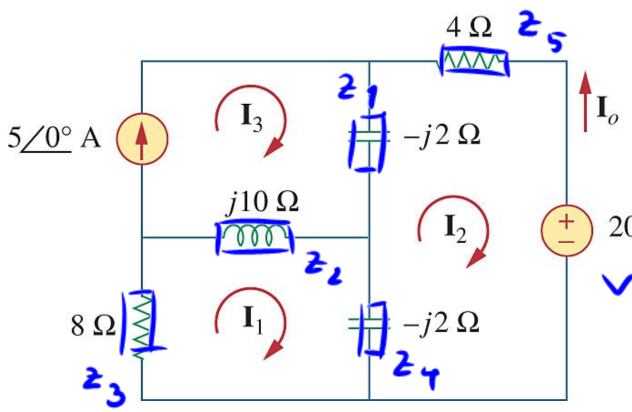


EXAMPLE 9.1.6. Determine current I_o in the circuit below using mesh analysis.

Mesh 3:

$I_3 = 5$

$I_o = -I_2$



$V_s = 20j$

Mesh 1:

$$-I_1 Z_3 - (I_1 - I_3) Z_2 - (I_1 - I_2) Z_4 = 0$$

Mesh 2:

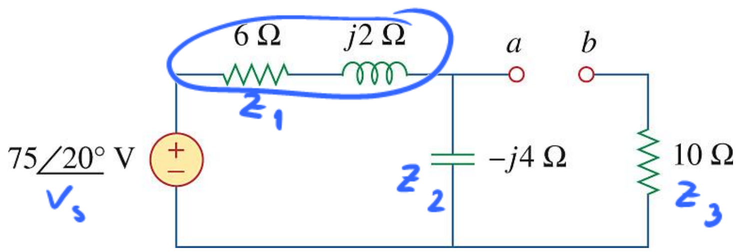
$$-(I_2 - I_1) Z_4 - (I_2 - I_3) Z_1 - I_2 \times Z_5 - V_s = 0$$

$\Rightarrow I_2 = 5 - 3.53j$

$I_o = -I_2 = -5 + 3.53j$

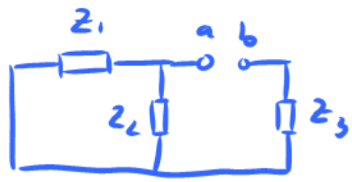
$= 6.12 \angle 144.8^\circ$

EXAMPLE 9.1.7. Find the Thevenin equivalent at terminals *a*-*b* of the circuit below.

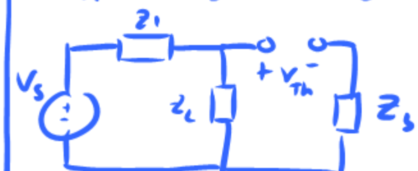


$V_s = 75\angle 20^\circ = 70.5 + 25.65j$
$Z_1 = 6 + 2j$
$Z_2 = -4j$
$Z_3 = 10$

$Z_{Th} = Z_3 + Z_1 /\!/ Z_2 = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2} = 12.4 - 3.2j$



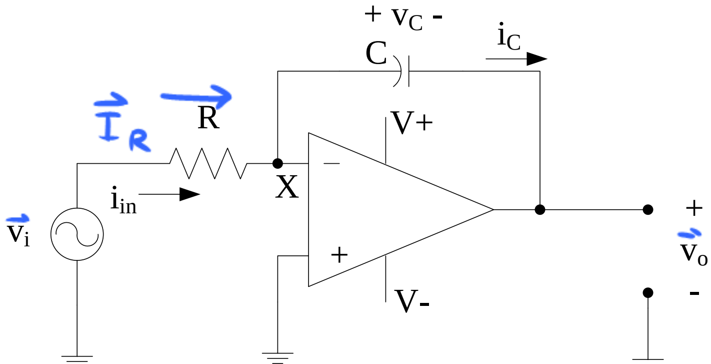
$V_{Th} = V_{Z_2} = \frac{Z_2}{Z_1 + Z_2} V_s = 29.5 - 37.2j$
$= 47.4\angle -51.6^\circ$



$v_{Th}(t) = 47.4 \cos(\omega t - 51.6^\circ)$

EXAMPLE 9.1.8. **Op Amp AC Circuits**: Find the (closed-loop) gain of the circuit below.

$\vec{I}_C = \vec{I}_R$



$\vec{V}_x = 0$

$\vec{I}_R = \frac{\vec{V}_i - \cancel{\vec{V}_x}^{\,0}}{R} = \frac{\vec{V}_i}{R} = \vec{I}_C = \frac{\cancel{\vec{V}_x}^{\,0} - \vec{V}_o}{1/j\omega C} = \frac{-\vec{V}_o}{1/j\omega C}$

$\vec{V}_o = -\frac{1}{j\omega C} \times \frac{\vec{V}_i}{R} = \frac{j\vec{V}_i}{\omega C R}$

$\frac{\vec{V}_o}{\vec{V}_i} = \boxed{\frac{j}{\omega C R}}$

9.2. Circuit With Multiple Sources Operating At Different Frequencies

A special care is needed if the circuit has multiple sources operating at different frequencies. In which case, one must add the responses due to the individual frequencies in the time domain. In other words, the superposition still works but

(a) We must have a different frequency-domain circuit for each frequency.

(b) The total response must be obtained by adding the individual response in the time domain.

9.2.1. Since the impedance depend on frequency, it is incorrect to try to add the responses in the phasor or frequency domain. To see this note that the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency ω. In particular, although

$$\sum_i V_{mi}\cos(\omega t + \phi_i) = \sum_i \mathrm{Re}\left\{\mathbf{V}_i e^{j\omega t}\right\} = \mathrm{Re}\left\{\left(\sum_i \mathbf{V}_i\right) e^{j\omega t}\right\},$$

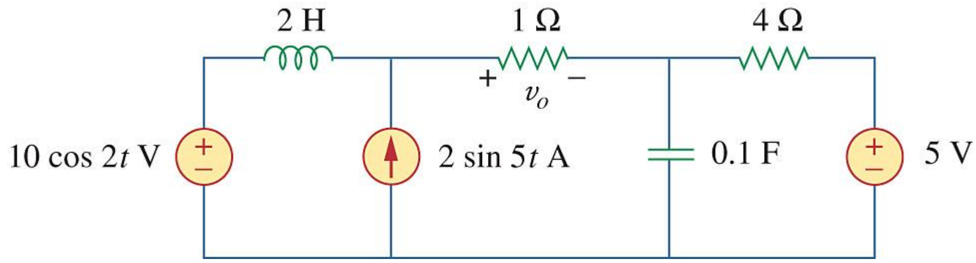
when we allow ω to be different for each sinusoid, generally

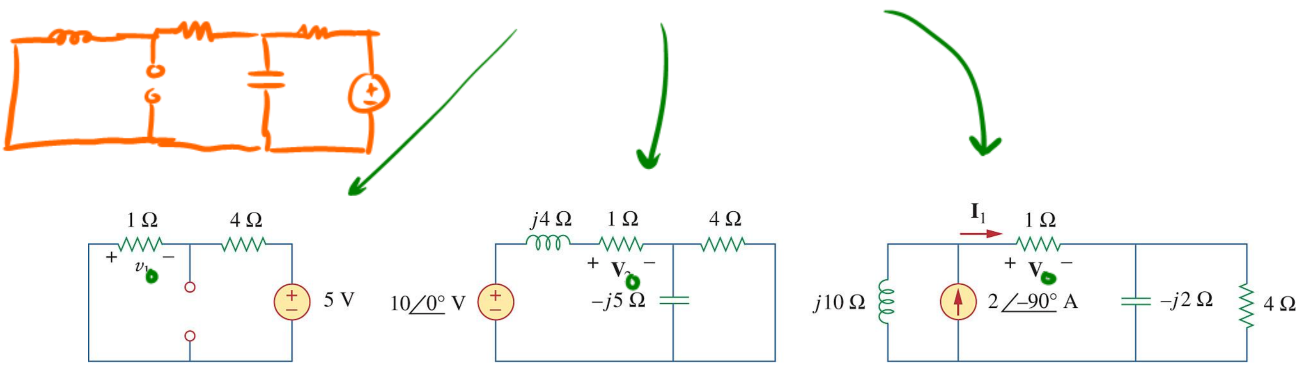
$$\sum_i V_{mi}\cos(\omega_i t + \phi_i) = \sum_i \mathrm{Re}\left\{\mathbf{V}_i e^{j\omega_i t}\right\} \neq \mathrm{Re}\left\{\left(\sum_i \mathbf{V}_i\right) e^{j\omega_i t}\right\}.$$

Therefore, it does not make sense to add responses at different frequencies in the phasor domain.

9.2.2. The Thevenin or Norton equivalent circuit (if needed) must be determined at each frequency and we have one equivalent circuit for each frequency.

EXAMPLE 9.2.3. Find v_o in the circuit below using the superposition theorem.





(a) (b) (c)

$\vec{V}_o = -1\ V$

$v_o(t) = -1\cos(0t+0)$
$= -1$

$\vec{V}_o = 2.5\angle -30.78^\circ\ V$

$v_o(t) = 2.5\cos(2t - 30.78^\circ)$

$\vec{V}_o = 2.33\angle -77.9^\circ\ V$

$v_o(t) = 2.33\cos(5t - 77.9^\circ)$

$v_o(t) = -1 + 2.5\cos(2t - 30.78^\circ) + 2.33\cos(5t - 77.9^\circ)$

CHAPTER 10

AC Power Analysis

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. The major concern in this chapter is power analysis.

10.0.4. Power is the most important quantity in electric utilities, electronic and communication systems because such systems involve transmission of power (or energy) from one point to another.

Every industrial and household electrical device (every fan, motor, lamp, pressing iron, TV, personal computer) has a **power rating** that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance.

10.0.5. The most common form of electric power is 50-Hz (Thailand) or 60-Hz (United States) ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

10.1. Instantaneous Power

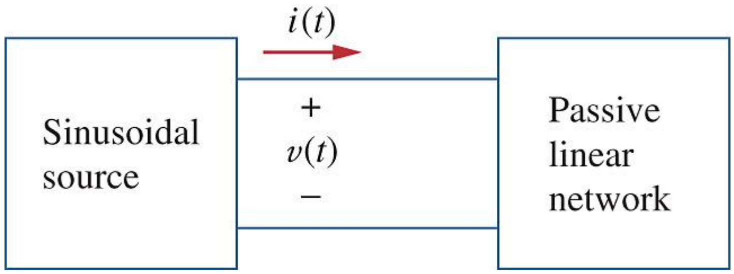
DEFINITION 10.1.1. The **instantaneous power** $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it.

Assuming the **passive sign convention**,

$$p(t) = v(t)i(t).$$

The instantaneous power is the power at any instant of time. It is **the rate at which an element absorbs energy**.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation.



Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v)$$

and

$$i(t) = I_m \cos(\omega t + \theta_i)$$

where V_m and I_m are the amplitudes, and θ_v and θ_i are the phase of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$\begin{aligned} p(t) = v(t)i(t) &= V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \\ &= V_m I_m \frac{e^{j(\omega t+\theta_v)} + e^{-j(\omega t+\theta_v)}}{2}\frac{e^{j(\omega t+\theta_i)} + e^{-j(\omega t+\theta_i)}}{2} \\ &= V_m I_m \frac{1}{4}\left(e^{j(2\omega t+\theta_v+\theta_i)} + e^{j(\theta_i-\theta_v)} + e^{j(\theta_v-\theta_i)} + e^{-j(2\omega t+\theta_v+\theta_i)}\right) \\ &= V_m I_m \frac{1}{2}\left(\frac{e^{j(\theta_v-\theta_i)} + e^{j(\theta_i-\theta_v)}}{2} + \frac{e^{j(2\omega t+\theta_v+\theta_i)} + e^{-j(2\omega t+\theta_v+\theta_i)}}{2}\right) \\ &= V_m I_m \frac{1}{2}\left(\cos\left(\theta_v - \theta_i\right) + \cos\left(2\omega t + \theta_v + \theta_i\right)\right). \end{aligned}$$

Alternatively, we can apply the trigonometric identity

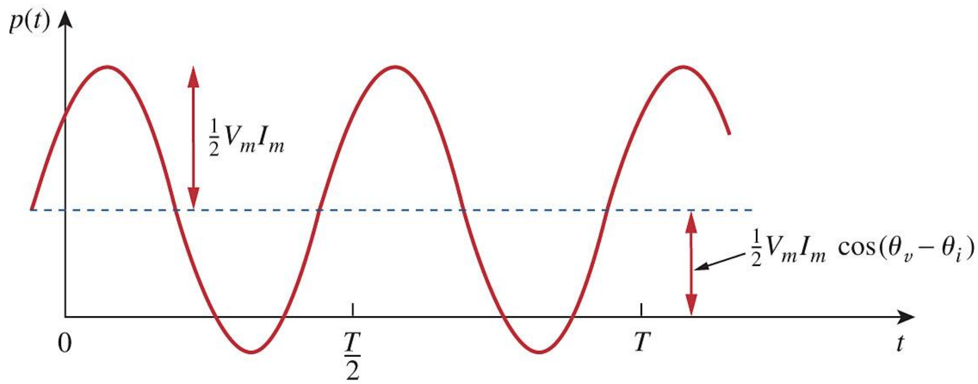
$$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

to directly arrive at the same result which is

$$p(t) = \underbrace{\frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}_{\text{constant term}} + \underbrace{\frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)}_{\text{time-dependent term}}.$$

This shows that the instantaneous power has two parts.

- **First Part**: a constant or time-independent term. Its value depends on the phase difference between the voltage and the current.
- **Second Part**: a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.



10.1.2. Consider the sketch of $p(t)$, we observe that

(a) $p(t)$ is periodic and has a period of $T_o = \frac{T}{2}$, where $T = \frac{2\pi}{\omega}$ is the period of the voltage and the current

(b) $p(t)$ may become positive for some part of each cycle and negative for the rest of the cycle.
- When $p(t)$ is positive, power is absorbed by the circuit.
- When $p(t)$ is negative, power is absorbed by the source.
  - In this case, power is transferred from the circuit to the source.
  - This is possible because of the storage elements (capacitors and inductors) in the circuit.

10.2. Average Power

The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure.

DEFINITION 10.2.1. **The average power is the average of the instantaneous power over one period.**

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i).$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and the current. Note that $p(t)$ is time varying while P does not depend on time.

10.2.2. Using the phasor forms of $v(t)$ and $i(t)$, which are $\mathbf{V} = V_m\angle\theta_v$ and $\mathbf{I} = I_m\angle\theta_i$, we obtain

$$\boxed{P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}\mathbf{Re}\{\mathbf{V}\mathbf{I}^*\}.}$$

10.2.3. Two special cases:

Case 1: When $\theta_v = \theta_i$, the voltage and the current are in phase. This implies a **purely resistive circui**t or resistive load R, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R,$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. This shows that **a purely resistive circuit (e.g. resistive load (R))absorbs power all times**.

Case 2: When $\theta_v - \theta_i = \pm 90°$, we have a **purely reactive circuit**, and

$$P = \frac{1}{2}V_m I_m \cos(90°) = 0$$

showing that **a purely reactive circuit (e.g. a reactive load L or C) absorbs no average power**.