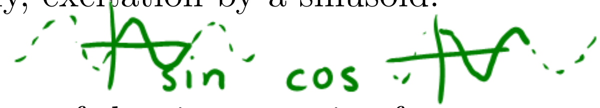
ECS 303 - Part 3A
Dr.Prapun Suksompong

CHAPTER 8

Sinusoids and Phasors

We now begins the analysis of circuits in which the voltage or current sources are time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a sinusoid.



8.0.1. Some terminology:

(a) A **sinusoid** is a signal that has the form of the sine or cosine function.
   - Turn out that you can express them all under the same notation using only cosine (or only sine) function.

(b) A sinusoidal current is referred to as **alternating current (AC)**.

(c) Circuits driven by sinusoidal current or voltage sources are called **AC circuits**.

$\phi = 0$

$x(t) = X_m \cos(\omega t)$

8.1. Sinusoids

≥ 0

8.1.1. Consider the sinusoidal signal (in cosine form)

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

$\omega = 2\pi f$

$f = \omega / 2\pi$

$\frac{2\pi}{\omega} = \frac{1}{f} = T$ (sec)

2π (rad)

$360°$ (degree)

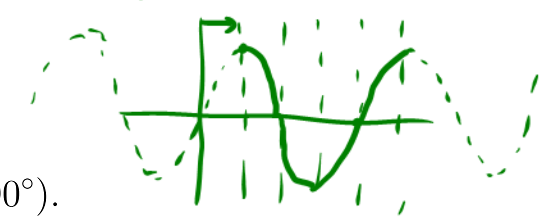
where

X_m: the amplitude of the sinusoid,
ω: the angular frequency in radians/s (or rad/s),
ϕ: the phase.

- When the signal is given in the sine form, it can be converted into its cosine form via the identity

$\phi = -90°$

$$\sin(x) = \cos(x - 90°).$$

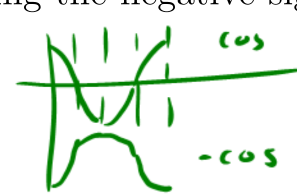


In particular,

$$X_m \sin(\omega t + \phi) = X_m \cos(\omega t + \phi - 90°).$$

- X_m is always non-negative. We can avoid having the negative sign by the following conversion:

$$-\cos(x) = \cos(x \pm 180°).$$



In particular,

$$-X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi \pm 180^\circ).$$

8.1.2. The **period** (the time of one complete cycle) of the sinusoid is

$$T = \frac{2\pi}{\omega}.$$

The unit of the period is in second if the angular frequency unit is in radian per second.

The **frequency** f (the number of cycles per second or hertz (Hz)) is the reciprocal of this quantity, i.e.,

$$f = \frac{1}{T}.$$

8.2. Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. The tradoff is that phasors are complex-valued.

8.2.1. The idea of phasor representation is based on Euler's identity:

$$e^{j\phi} = \cos\phi + j \sin\phi,$$

From the identity, we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$:

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\}, \quad \sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\},$$

where Re and Im stand for the real part of and the imaginary part of $e^{j\phi}$.

8.2.2. A phasor is a complex number that represents the amplitude and phase of a sinusoid. Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, then

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}\left\{V_m e^{j(\omega t + \phi)}\right\} = \mathrm{Re}\left\{V_m e^{j\phi} \cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathbf{V} e^{j\omega t}\right\},$$

where

phasor

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi.$$

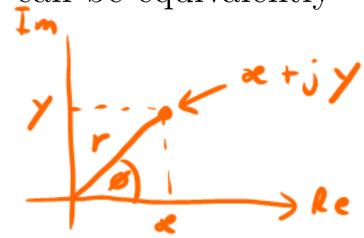
$\mathbf{V}$ is called the ***phasor representation*** of the sinusoid $v(t)$. In other words, a phasor is a complex number that represents amplitude and phase of a sinusoid.

8.2.3. Remarks:

- Whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is **implicitly** present. It is therefore important, when dealing with phasors, to keep in mind the (angular) frequency ω of the phasor.

- To obtain the sinusoid corresponding to a given phasor $\mathbf{V}$, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part.

  Equivalently, given a phasor, we obtain the time-domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor.
- Any complex number z (including any phasor) can be equivalently represented in three forms.
  (a) Rectangular form: $z = x + jy$.
  (b) Polar form: $z = r\angle\phi$.
  (c) Exponential form: $z = re^{j\phi}$

  where the relations between them are

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x}.$$

$$x = r\cos\phi, \quad y = r\sin\phi.$$

  As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form.

8.2.4. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m\cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m\angle\phi.$$

Time domain representation ⇔ Phasor domain representation

Example 8.2.5. Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40°)$ A $\Rightarrow 6\angle{-40°}$
(b) $v = -4\sin(30t + 50°)$ V $= -4\cos(30t + 50° - 90°) = -4\cos(30t - 40°)$

$= 4\cos(30t - 40° + 180°) = 4\cos(30t + 140°)$

Example 8.2.6. Find the sinusoids represented by these phasors: $\Rightarrow 4\angle 140°$

(a) $\mathbf{I} = -3 + j4$ A $= 5\angle 127°$ A

$\sqrt{(-3)^2 + 4^2}$ $\quad \tan^{-1}\left(\frac{4}{-3}\right) = 127°$ $\quad 5\cos(\omega t + 127°)$

(b) $\mathbf{V} = j8e^{-j20°}$ V

$e^{j90°} \times 8 \times e^{-j20°} = 8e^{j70°} = 8\angle 70° \to 8\cos(\omega t + 70°)$

8.2.7. The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

(a) $v(t)$ is the instantaneous or time-domain representation, while V is the frequency or phasor-domain representation.
(b) $v(t)$ is time dependent, while V is not.

↑ simply a complex number

(c) $v(t)$ is always real with no complex term, while V is generally complex.

8.2.8. Adding sinusoids of the *same frequency* is equivalent to adding their corresponding phasors. To see this,

$A_1 e^{j\phi_1}$

$$A_1\cos(\omega t+\phi_1)+A_2\cos(\omega t+\phi_2) = \operatorname{Re}\left\{\mathbf{A}_1 e^{j\omega t}\right\}+\operatorname{Re}\left\{\mathbf{A}_2 e^{j\omega t}\right\}$$
$$= \operatorname{Re}\left\{(\mathbf{A}_1+\mathbf{A}_2)\, e^{j\omega t}\right\}.$$

$\vec{A}_1+\vec{A}_2$

$=\vec{B}$

$=B\angle\theta$

$\downarrow$

$B\cos(\omega t+\theta)$

8.2.9. Properties involving differentiation and integration:

(a) **Differentiating** a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$. In other words,

$$\frac{dv(t)}{dt} \Leftrightarrow j\omega\mathbf{V}.$$

$\boxed{y(t) = \frac{d}{dt}v(t) \;\downarrow\; \vec{Y} = j\omega\vec{V}}$

To see this, suppose $v(t)=V_m\cos(\omega t+\phi)$. Then,

$v(t)=\operatorname{Re}\{\vec{V}e^{j\omega t}\}$

$$\frac{dv}{dt}(t) = -\omega V_m\sin(\omega t+\phi) = \omega V_m\cos(\omega t+\phi-90^\circ+180^\circ) = \omega V_m \underbrace{\cos(\omega t+\phi+90^\circ)}_{}$$

$\operatorname{Re}\{e^{j(\omega t+\phi+90^\circ)}\}$

$\frac{d}{dt}v(t) =$ $\vec{V}$

$$= \operatorname{Re}\left\{\omega V_m e^{j\phi} e^{j90^\circ}\cdot e^{j\omega t}\right\} = \operatorname{Re}\left\{j\omega\mathbf{V}e^{j\omega t}\right\}$$

j

$\operatorname{Re}\{\frac{d}{dt}\vec{V}e^{j\omega t}\}$

(b) **Integrating** a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$. In other words,

$$\int v(t)dt \Leftrightarrow \frac{\mathbf{V}}{j\omega}.$$

$y(t)=\int v(t)dt$

$\frac{d}{dt}y(t) = v(t)$

$j\omega\times\vec{Y} = \vec{V}$

$\vec{Y} = \frac{1}{j\omega}\vec{V}$

EXAMPLE 8.2.10. Find the voltage $v(t)$ in a circuit described by the intergrodifferential equation

$\omega = 5$

$$2\frac{dv}{dt}+5v+10\int vdt = \overset{5}{\cancel{2}}0\cos(5t-30^\circ)$$

using the phasor approach.

$2\times j\omega\vec{V} + 5\vec{V} + 10\frac{\vec{V}}{j\omega} = 50\angle-30^\circ$

$10j\vec{V} + 5\vec{V} - 2j\vec{V} = 50\angle-30^\circ$

$(5+8j)\vec{V} = 50\angle-30^\circ$

$\vec{V} = \frac{50\angle-30^\circ}{5+8j} = 0.2-5.3j = 5.3\angle-88^\circ$

$\boxed{5.3\cos(5t-88^\circ)}$

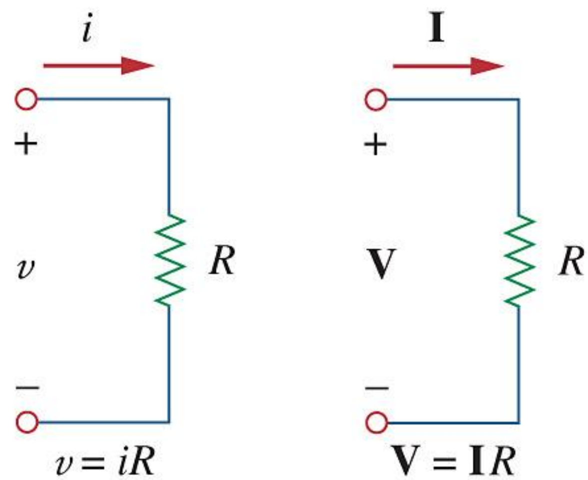
8.3. Phasor relationships for circuit elements

8.3.1. Resistor R: If the current through a resistor R is

$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = i(t)R = RI_m \cos(\omega t + \phi).$$



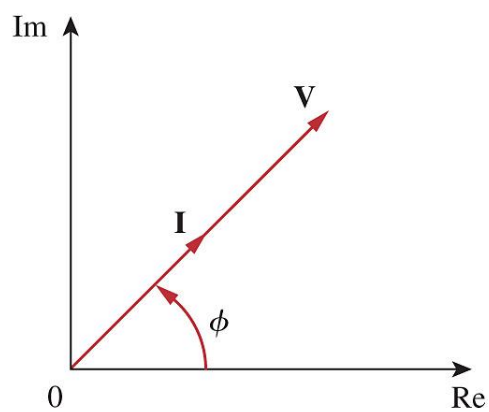
The phasor of the voltage is

$$\mathbf{V} = RI_m \angle \phi.$$

Hence,

$$\mathbf{V} = \mathbf{I}R.$$

We note that voltage and current are **in phase** and that the voltage-current relation for the resistor in the phasor domain continues to be Ohms law, as in the time domain.

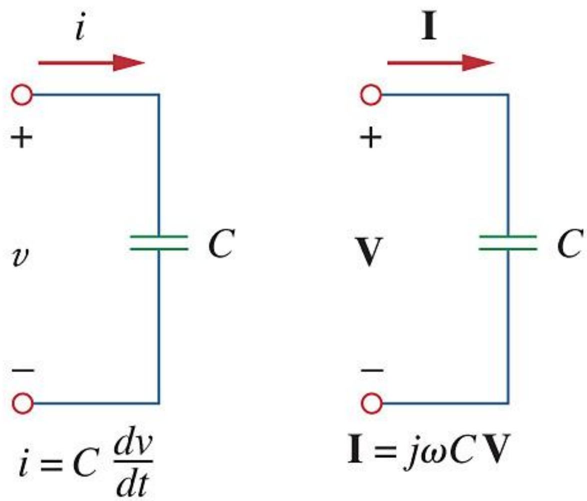


8.3.2. Capacitor C: If the voltage across a capacitor C is

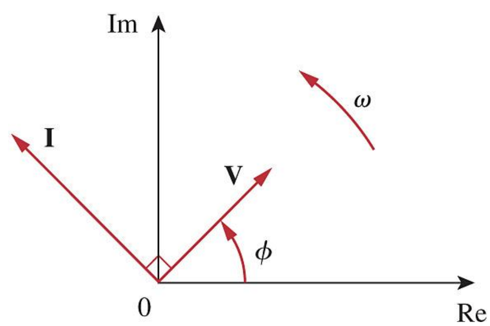
$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle \phi,$$

the current through it is given by

$$i(t) = C\frac{dv(t)}{dt} \Leftrightarrow \mathbf{I} = j\omega C\mathbf{V} = \omega C V_m \angle(\phi + 90^\circ).$$



The voltage and current are 90° out of phase. Specifically, the current leads the voltage by 90°.



- Mnemonic: CIVIL

  In a Capacitive (C) circuit, I leads V. In an inductive (L) circuit, V leads ~~V~~.

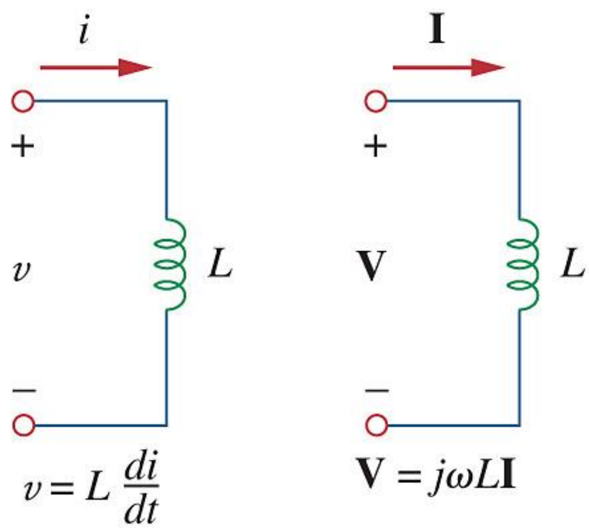
  I

8.3.3. Inductor L: If the current through an inductor L is

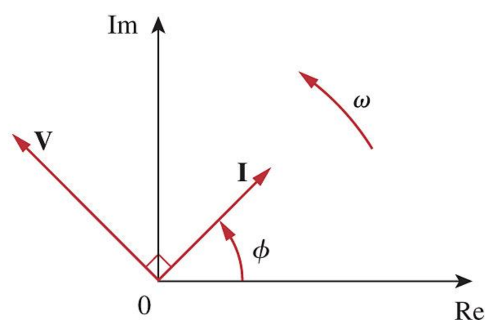
$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = L\frac{di(t)}{dt} \Leftrightarrow \mathbf{V} = j\omega L\mathbf{I} = \omega L I_m \angle(\phi + 90^\circ).$$



The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°.



Element	Time domain	Frequency domain	
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	$= (j\omega L)\,\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$	$= \left(\dfrac{1}{j\omega C}\right)\mathbf{I}$

8.4. Impedance and Admittance

Thus, we obtained the voltage current relations for the three passive elements as

$$\mathbf{V} = \mathbf{I}R, \quad \mathbf{V} = j\omega L\mathbf{I}, \quad \mathbf{I} = j\omega C\mathbf{V}.$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor of current as

$$\frac{\mathbf{V}}{\mathbf{I}} = \overset{Z}{R}, \quad \frac{\mathbf{V}}{\mathbf{I}} = \overset{Z}{j\omega L}, \quad \frac{\mathbf{V}}{\mathbf{I}} = \overset{Z}{\frac{1}{j\omega C}}. = -j\frac{1}{\omega C}$$

From these equations, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{IZ}. \leftarrow \text{Ohm's law.}$$

Definition 8.4.1. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta,$$

with

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R} \pm 180^\circ, \quad R = |\mathbf{Z}|\cos\theta, \quad X = |\mathbf{Z}|\sin\theta.$$

$R = \mathrm{Re}\{\mathbf{Z}\}$ is called the **resistance** and $X = \mathrm{Im}\{\mathbf{Z}\}$ is called the **reactance**.

$Z = 5 + 3j$

The reactance X may be positive or negative. We say that the impedance is **inductive** when X is positive or **capacitive** when X is negative. $Z = 5 - 3j$

Definition 8.4.2. The **admittance** ($\mathbf{Y}$) is the reciprocal of impedance, measured in Siemens (S). The admittance of an element(or a circuit) is the ratio of the phasor current through it to phasor voltage across it, or

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}.$$

8.4.3. Kirchhoff's laws (KCL and KVL) hold in the phasor form.

To see this, suppose $v_1, v_2, \ldots, v_n$ are the voltages around a closed loop, then

$$v_1 + v_2 + \cdots + v_n = 0.$$

If each voltage v_i is a sinusoid, i.e.

$$v_i = V_{mi} \cos(\omega t + \phi_i) = \mathrm{Re}\left\{\vec{V}_i e^{j\omega t}\right\}$$

with phasor $\mathbf{V}_i = V_{mi}\angle\phi_i = V_{mi}e^{j\phi_i}$, then

$$\mathrm{Re}\left\{\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right) e^{j\omega t}\right\} = 0,$$

which must be true for all time t. To satisfy this, we need

Remember : $\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0.$

Hence, KVL holds for phasors. around loop.

If $\mathrm{Re}\left\{\vec{Z} e^{j\omega t}\right\} = 0$ then $\vec{Z} = 0$

Proof

$\mathrm{Re}\left\{|\vec{Z}| e^{j\phi_z} e^{j\omega t}\right\}$

$= \mathrm{Re}\left\{|\vec{Z}| e^{j(\phi_z + \omega t)}\right\}$

$= |\vec{Z}| \cos(\phi_z + \omega t) = 0$

$\Rightarrow |\vec{Z}| = 0$

Similarly, we can show that KCL holds in the frequency domain, i.e., if the currents $i_1, i_2, \ldots, i_n$ be the currents entering or leaving a closed surface at time t, then

$$i_1 + i_2 + \cdots + i_n = 0.$$

If the currents are sinusoids and $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are their phasor forms, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0.$$

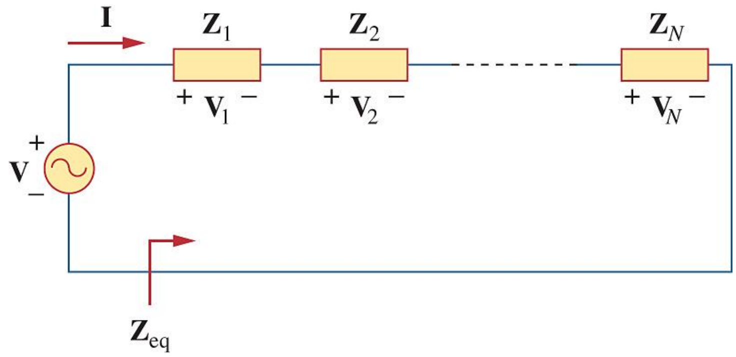
Major Implication: Since Ohm's Law and Kirchhoff's Laws hold in phasor domain, **all resistance combination, analysis methods** (nodal and mesh analysis) **and circuit theorems** (linearity, superposition, source transformation, and Thevenin's and Norton's equivalent circuits) that we have previously studied for dc circuits **apply to ac circuits**!!!

Just think of impedance as a complex-valued resistance!!

In addition, our ac circuits can now effortlessly include capacitors and inductors which can be considered as impedances whose values depend on the frequency ω of the ac sources!!

8.5. Impedance Combinations

Consider N series-connected impedances as shown below.



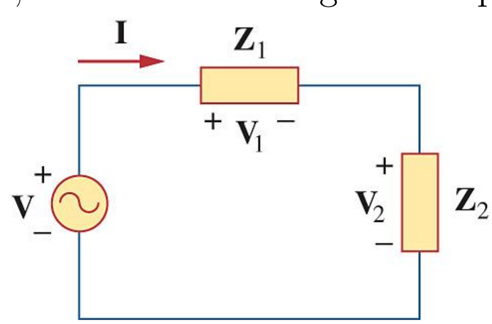
The same current **I** flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N.$$

In particular, if $N = 2$, the current through the impedance is



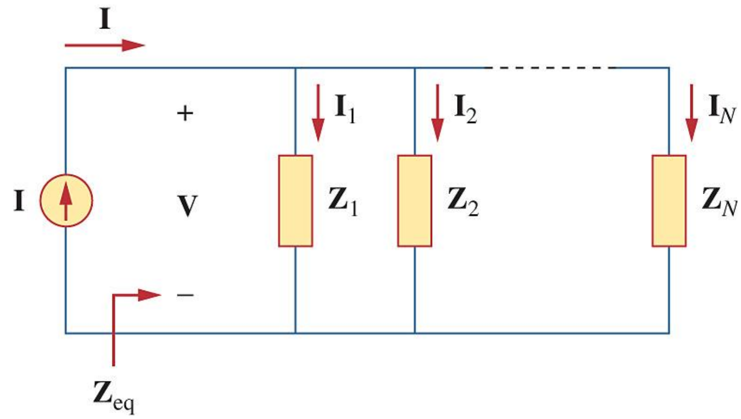
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$,

$$\mathbf{V_1} = \frac{\mathbf{Z_1}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{V}, \quad \mathbf{V_2} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{V}$$

which is the **voltage-division** relationship.

Now, consider N parallel-connected impedances as shown below.



The voltage across each impedance is the same. Applying KCL at the top node gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right).$$

The equivalent impedance $\mathbf{Z}_{eq}$ can be found from

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}.$$

When $N = 2$,

$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

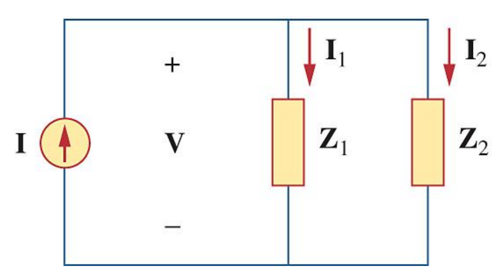
Because

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2,$$

we have

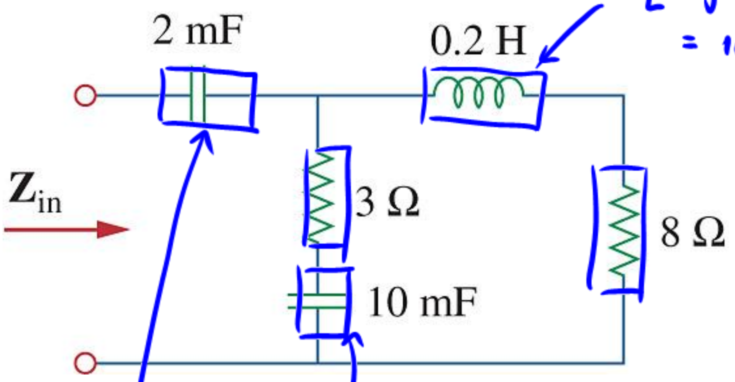
$$\mathbf{I_1} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}, \quad \mathbf{I_2} = \frac{\mathbf{Z_1}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}$$

which is the **current-division** principle.



EXAMPLE 8.5.1. Find the input impedance of the circuit below. Assume that the circuit operates at $\omega = 50$ rad/s.

$Z_L = j\omega L = j\times 50\times 0.2$
$= 10j$



$\frac{1000}{j\times 50\times 2} = \frac{1}{j\omega C} = Z_{c_2}$

$-10j =$

$Z_{c_1} = \frac{1}{j\omega C} = \frac{1000}{j\times 50\times 10} = -2j$

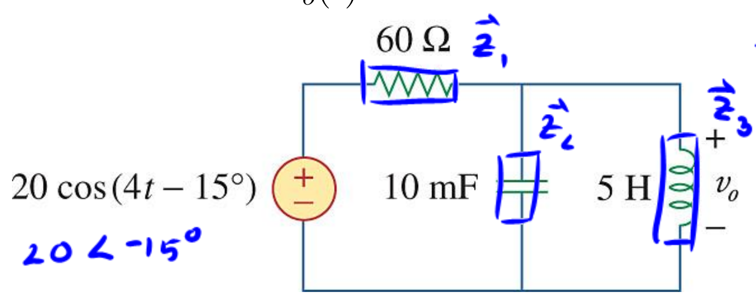
$\vec{Z}_{in} = Z_{c2} + (3 + Z_{c_1}) // (8 + Z_L)$

$= -10j + (3-2j) // (8+10j) = -10j + \frac{(3-2j)(8+10j)}{11+8j}$

$= 3.22 - 11j$

$= 11.53 \angle -73^\circ$

EXAMPLE 8.5.2. Determine $v_o(t)$ in the circuit below.



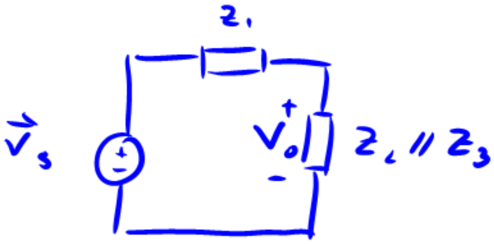
$20 \angle -15^\circ$

$\vec{V}_s = 19.3 - 5.2j$

$Z_1 = 60$

$Z_2 = \frac{1}{j\omega C} = \frac{1000}{j4\times 10}$
$= -25j$

$Z_3 = j\omega L = j\times 4\times 5$
$= 20j$



$V_o = \frac{Z_2 // Z_3}{(Z_2 // Z_3) + Z_1} \times V_s = 17.15 \angle 15.96^\circ$

$v_o(t) = 17.15 \cos(4t + 15.96^\circ)$

CHAPTER 9

Sinusoidal Steady State Analysis

In the previous chapter, we have learned that the steady state response of a circuit to sinusoidal inputs can be obtained by using phasors. In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, superposition, and source transformations are applied in analyzing ac circuits.

Steps to analyze ac circuits, using phasor domain:

Step 1. Transform the circuit to the phasor or frequency domain.
- Not necessary if the problem is specified in the frequency domain.

Step 2. Solve the problem using circuit techniques (e.g., nodal analysis, mesh analysis, Thevenin's theorem, superposition, or source transformations)
- The analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Step 3. Transform the resulting phasor back to the time domain.