

ECS 303 - Part 2D

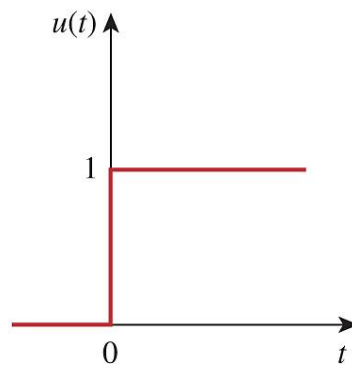
Dr. Prapun Sukksompong

7.3. Unit step function

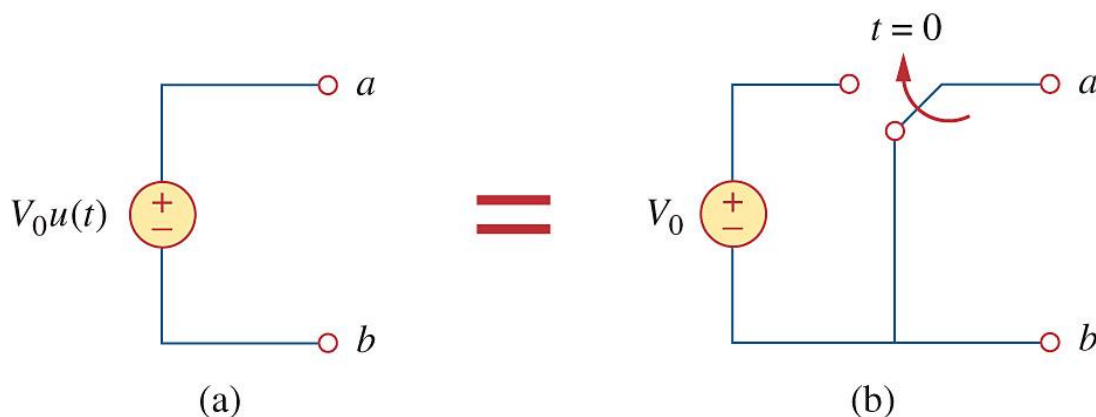
We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers.

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t . In mathematical terms,

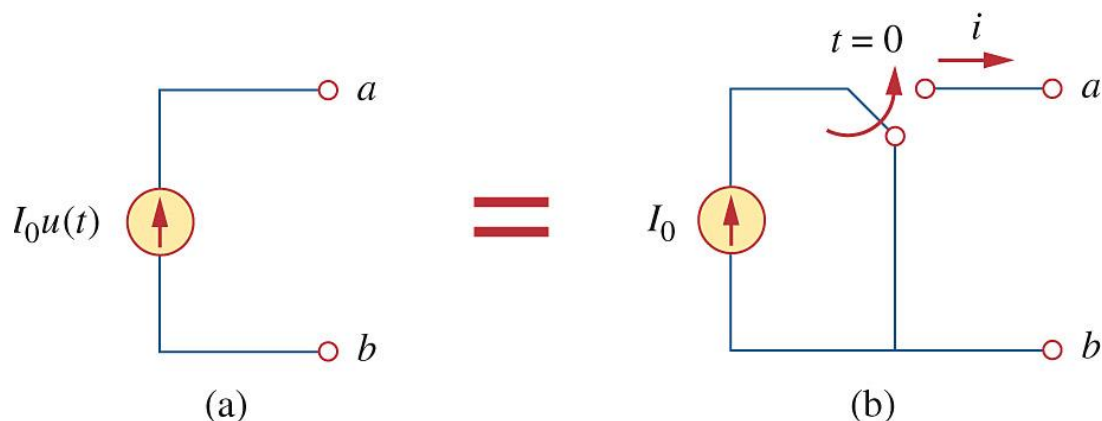
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



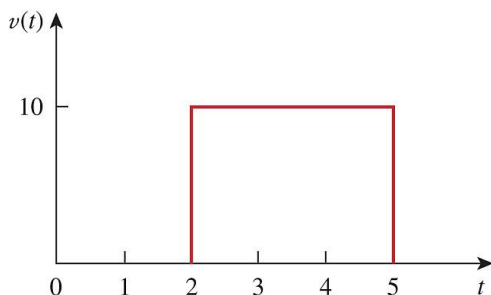
For example, here is a voltage source of $V_0u(t)$



Similarly, here is a current source of $I_0u(t)$

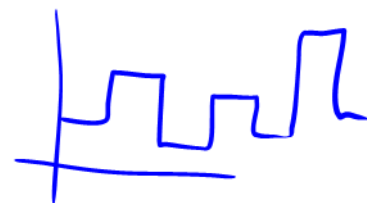
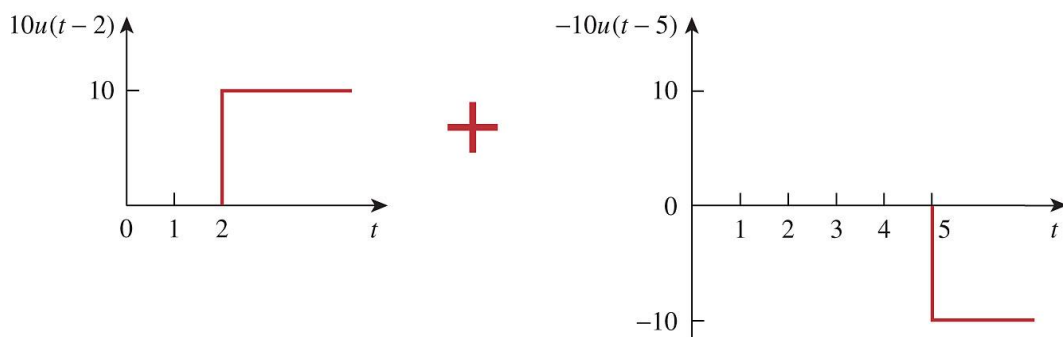


We can also use the step functions to represent the gate function (pulse) which may be regarded as a step function that switches on at one value of t and switches off at another value of t . The gate function below switches on at $t = 2$ s and switches off at $t = 5$ s.



It consists of the sum of two unit step functions:

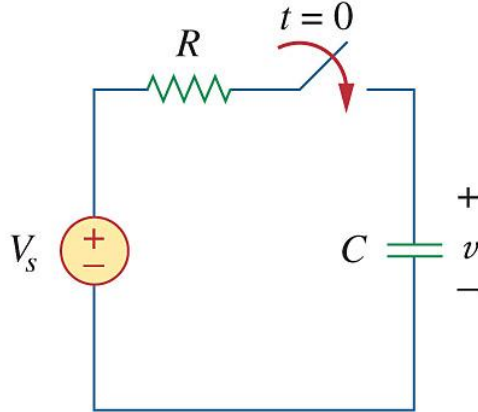
$$v(t) = 10u(t - 2) - 10u(t - 5).$$



When the dc source of an RC circuit is suddenly applied (i.e., this happens when the capacitor is being charged), the voltage or current source can be modeled as a step function, and the response is known as a **step response**.

7.4. Step Response of an RC Circuit

Consider an RC circuit with voltage step input below:



The voltage across the capacitor is v . We assume an initial voltage V_0 on the capacitor. Since the voltage of a capacitor cannot change instantaneously $v(0^-) = v(0^+) = V_0$, where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching.

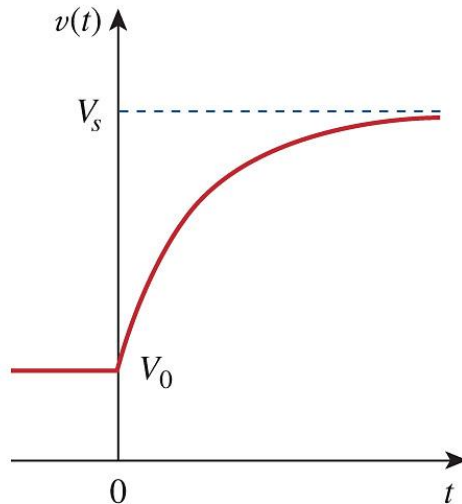
For $t > 0$, applying KCL, we have

$$C \frac{dv}{dt} + \frac{v - V_s}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

The solution is

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t \geq 0.$$



In general, the step response of an RC circuit whose switch changes position at time $t = 0$ can be written as

$$v(t) = v(\infty) + (v(0) - v(\infty))e^{\frac{-t}{\tau}}, \quad t > 0.$$

where $v(0)$ is the initial voltage and $v(\infty)$ is the final or steady-state value. We obtain

- (a) $v(0)$ from the given circuit for $t < 0$ and
- (b) $v(\infty)$ and τ from the circuit for $t > 0$.

Remark: If the switch changes position at time $t = t_o$ instead of at $t = 0$, the response becomes

$$v(t) = v(\infty) + (v(t_o) - v(\infty))e^{\frac{-(t-t_o)}{\tau}}, \quad t > t_o$$

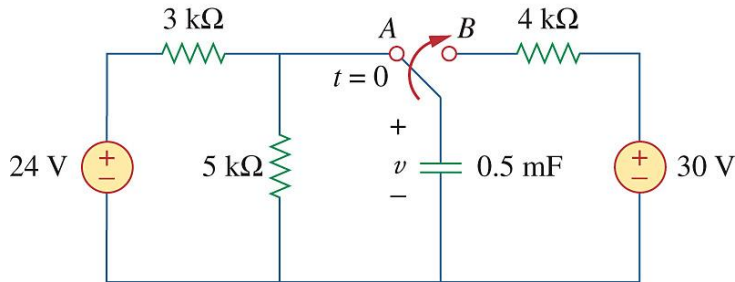
where $v(t_o)$ is the initial value at $t = t_o$.

Thus, the key in finding the step response of an RC circuit is to determine three values:

1. The initial capacitor voltage $v(0)$ or $V(t_0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ .

Keep in mind that this technique applies only for step responses.

Ex. The switch in the circuit below has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



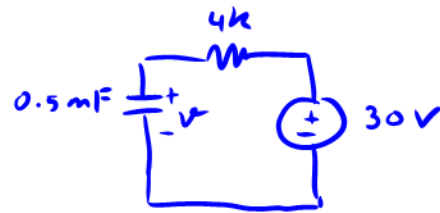
$$v(t) = v(\infty) + (v(0) - v(\infty)) e^{-t/\tau}$$

$$\tau = R_m \times C = 4k \times 0.5m = 2 \text{ sec.}$$

For $t < 0$



For $t > 0$

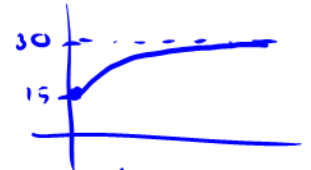


For capacitor, no jump in voltage

$$v(0) = v(0^-) = 15V$$

$$v(0) = 15V$$

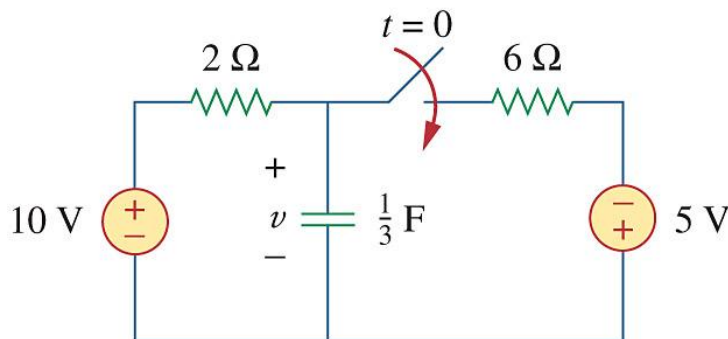
$$v(\infty) = 30V$$

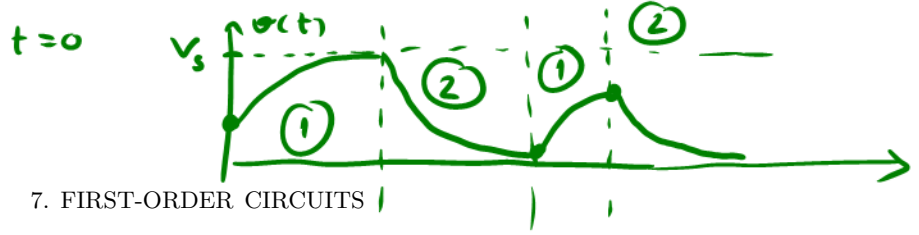


$$v(t) = 30 + (15 - 30) e^{-t/2} \text{ V, } t > 0$$

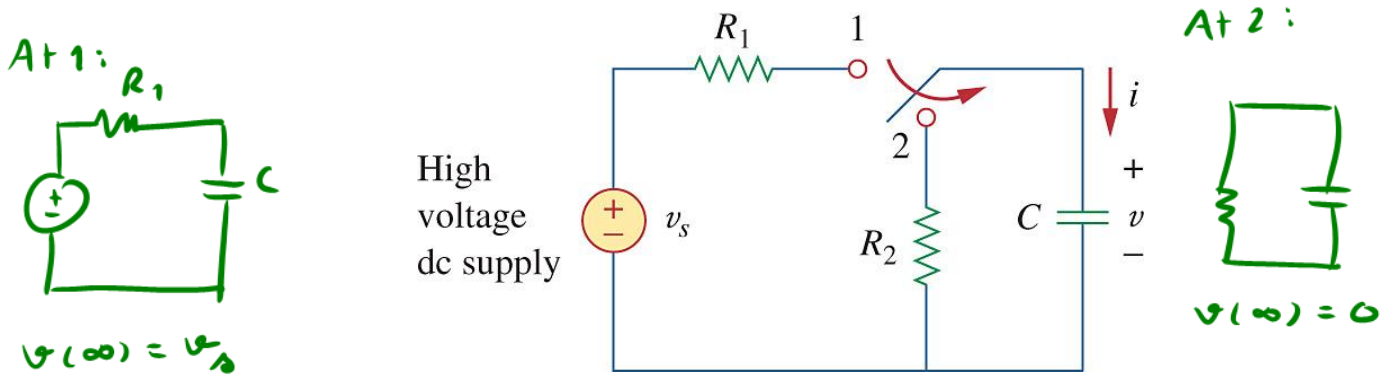
$$= 30 - 15 e^{-t/2} \text{ V, } t > 0$$

Ex. Find $v(t)$ for $t > 0$ in the circuit below. Assume the switch has been open for a long time and is closed at $t = 0$. Numerically evaluate $v(t)$ at $t = 0.5$.



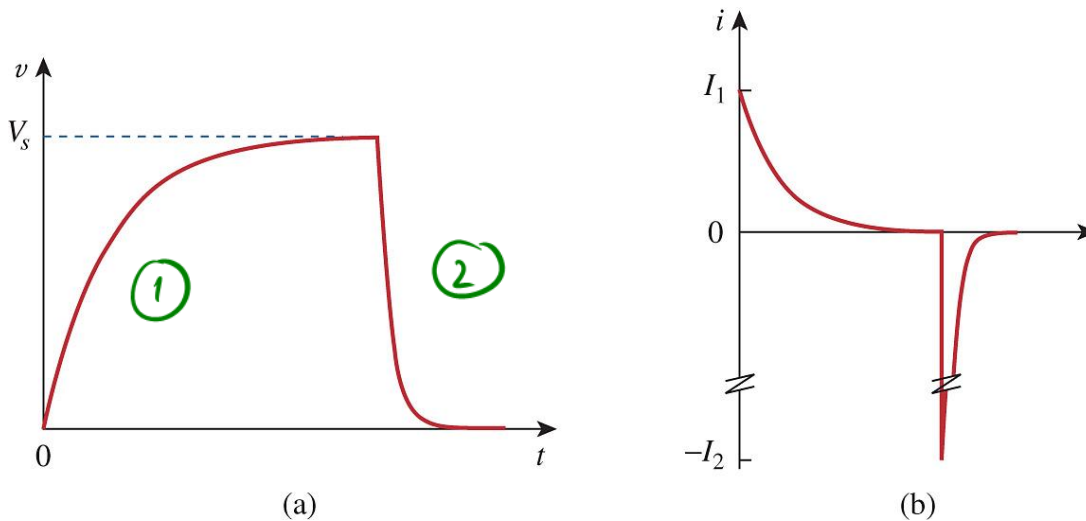


Ex. An electronic flash unit provides a common example of an RC circuit. This application exploits the ability of the capacitor to oppose any abrupt change in voltage. A simplified circuit is shown below. It consists essentially of a high-voltage dc supply, a current-limiting large resistor R_1 , and a capacitor C in parallel with the flashlamp of low resistance R_2 .

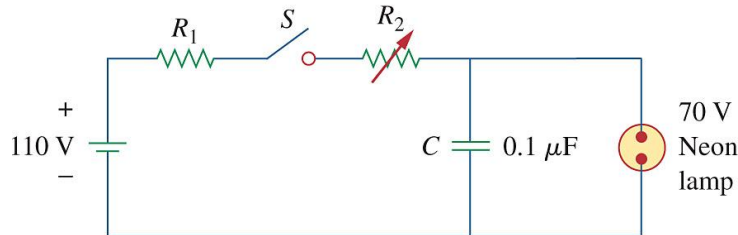


- When the switch is in position 1, the capacitor charges slowly due to the large time constant ($\tau = R_1C$). The capacitor voltage rises gradually from zero to V_s , while its current decreases gradually from $I_1 = V_s/R_1$ to zero.
- With the switch in position 2, the capacitor voltage is discharged. The low resistance R_2 of the photolamp permits a high discharge current with peak $I_2 = V_s/R_2$ in a short duration.

This simple RC circuit provides a short-duration, high current pulse. Such a circuit also finds applications in electric spot welding and the radar transmitter tube.



Ex. An RC circuit can be used to provide various time delays. Consider the circuit below. It basically consists of an RC circuit with the capacitor connected in parallel with a neon lamp. The voltage source can provide enough voltage to fire the lamp.



- When the switch is closed, the capacitor voltage increases gradually toward 110 V at a rate determined by the circuit's time constant, $(R_1 + R_2)C$. The lamp will act as an open circuit and not emit light until the voltage across it exceeds a particular level, say 70 V.
- When the voltage level is reached, the lamp fires (goes on), and the capacitor discharges through it. Due to the low resistance of the lamp when on, the capacitor voltage drops fast and the lamp turns off. The lamp acts again as an open circuit and the capacitor recharges.
- By adjusting R_2 , we can introduce either short or long time delays into the circuit and make the lamp fire, recharge, and fire repeatedly every time constant $\tau = (R_1 + R_2)C$, because it takes a time period τ to get the capacitor voltage high enough to fire or low enough to turn off.
- The warning blinkers commonly found on road construction sites are one example of the usefulness of such an RC delay circuit.

7.5. Step Response of an RL Circuit

Here, our goal is to find the inductor current i as the circuit response.

In general, the step response of an RL circuit whose switch changes position at time $t = 0$ can be written as

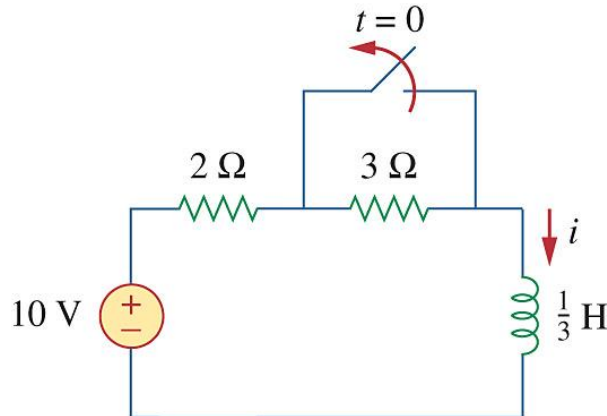
$$i(t) = i(\infty) + (i(0) - i(\infty))e^{-\frac{t}{\tau}}, \quad t > 0.$$

where $i(0)$ is the initial current and $i(\infty)$ is the final or steady-state value. If the switch changes position at time $t = t_0$ instead of at $t = 0$, the response becomes

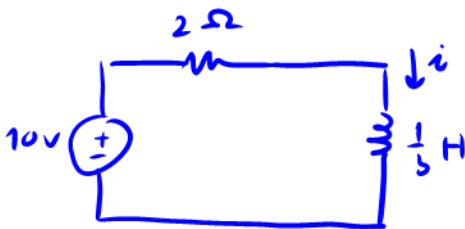
$$i(t) = i(\infty) + (i(t_0) - i(\infty))e^{-\frac{(t-t_0)}{\tau}}, \quad t > t_0$$

where $v(t_0)$ is the initial inductor current value at time $t = t_0$.

Ex. Find $i(t)$ in the circuit below for $t > 0$. Assume that the switch has been closed for a long time.



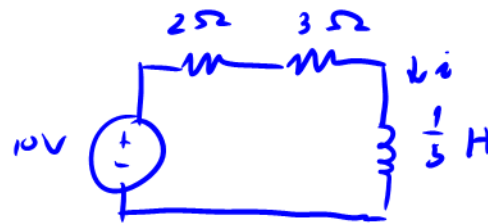
$t < 0$



$t > 0$

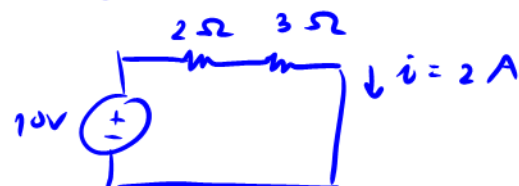
$$i(0) = i(0^-) = 5 \text{ A}.$$

$$\tau = \frac{L}{R_{th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}.$$



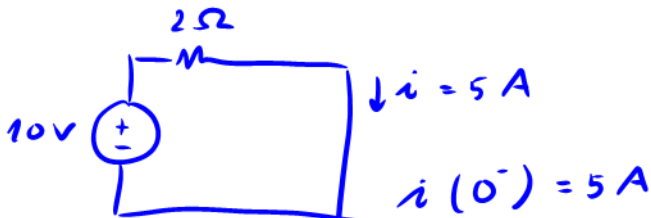
$$i(\infty) = 2 \text{ A}$$

$$i(t) = i(\infty) + (i(0) - i(\infty))$$



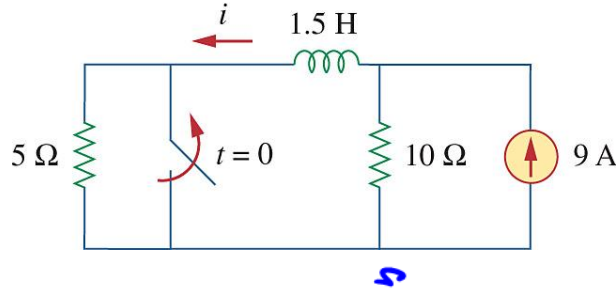
$$\times e^{-t/\tau}$$

$t = 0^-$



$$i(0^-) = 5 \text{ A}$$

Ex. The switch in the circuit below has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.



$$i(t) = \frac{30}{11} + \left(4 - \frac{30}{11}\right) e^{-(t-4) \times \frac{22}{15}}$$

Ex. At $t = 0$, switch 1 in the circuit below is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$.

$$\tau = \frac{L}{R} = \frac{5}{22/3} = \frac{15}{22}$$

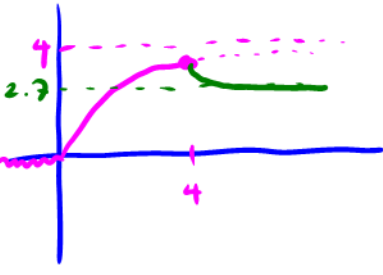
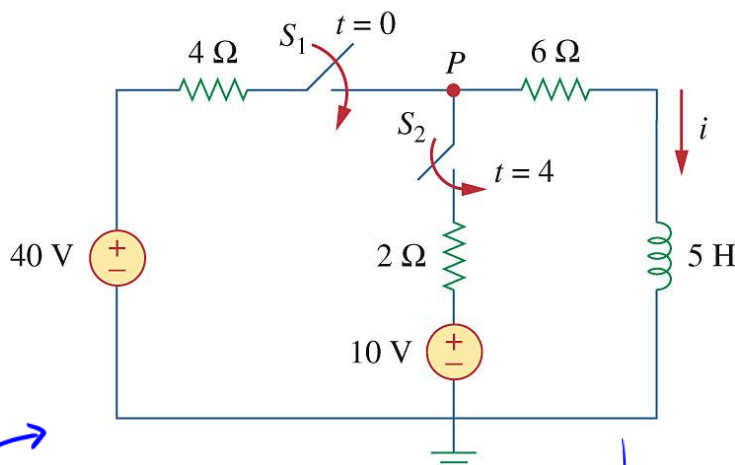
$$i(4) = 4$$

$$R_{th} = 4 \parallel 2 + 6 = \frac{22}{3}$$

$$\frac{V_x - 40}{4} + \frac{V_x - 10}{2} + \frac{V_x}{6} = 0$$

$$V_x = \frac{180}{11}$$

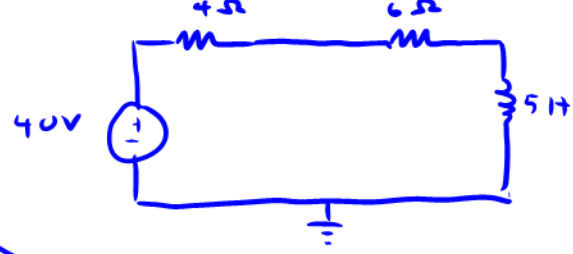
$$i(\infty) = \frac{180/11}{6} = \frac{30}{11}$$



Case 1: $t < 0$
 $i(\infty) = 0$
 $i(0^-) = 0$

No current-value jump for inductor

Case 2: $0 < t < 4$

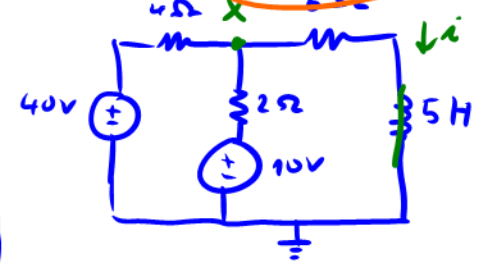


$$i(0) = 0$$

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A}$$

$$\tau = \frac{L}{R} = \frac{5}{10} = \frac{1}{2} \text{ sec.}$$

Case 3: $t > 4$



$$i(t) = i(\infty) + (i(t_0) - i(\infty)) e^{-(t-t_0)/\tau}$$

$$= 4 + (0 - 4) e^{-2t} \quad i(4)$$

$$= 4 - 4e^{-2t} \text{ A} \quad = 4 - 4e^{-8}$$