ECS 303 - Part 2C
Dr.Prapun Suksompong

CHAPTER 7

First-Order Circuits

In the previous chapter, we have learned about the relationships between current-voltage of capacitor and inductors, which are considered storage elements. These elements can be used to store energy and release energy when needed. In this chapter, we will see how the voltage or current behaves during the charging/discharging of these storage elements.

Specifically, we shall examine two types of simple circuits with a storage element:

(a) A circuit with a resistor and a capacitor (called an **RC circuit**); and

(b) a circuit with a resistor and an inductor (called an **RL circuit**).

These circuits may look simple but they find continual applications in electronics, communications, and control systems.

Applying Kirchhoff's laws to the RC and RL circuits produce **first order differential equations**. Hence, the circuits are collectively known as **first-order circuits**. There are two ways to excite the circuits.

(a) By initial conditions of the storage elements in the circuit.
   - Also known as *source-free circuits*
   - Assume that energy is initially stored in the capacitive or inductive element
   - This is the **discharging process**.

(b) By using independent sources
   - This is the **charging process**
   - For this chapter, we will consider independent dc sources.

Before we start our circuit analysis, it may be useful to first consider one mathematical fact:

The solution of the first-order differential equation

$$\frac{d}{dt}x(t) = a(t)x(t) + b(t)$$

is given by

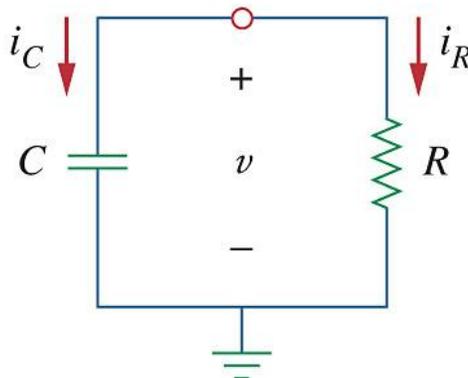$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \mu)b(\mu)d\mu,$$

where

$$\phi(t, s) = \exp\left(\int_{s}^{t} a(\mu)d\mu\right)$$

is called the integrating factor or the Green's function. In particular, when $a(t)$ is simply a constant a, then $\phi(t, s) = e^{a(t-s)}$ and

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\mu)}b(\mu)d\mu.$$

7.1. Source-Free RC Circuits

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistor.

Consider a series combination of a resistor an initially charged capacitor. We assume that at time $t = 0$, the initial voltage is $v(0) = V_o$. (Hence, the initial stored energy is $w(0) = \frac{1}{2}CV_o^2$.)

Applying KCL, we get

$$i_C + i_R = 0$$
$$C\frac{dv}{dt} + \frac{v}{R} = 0$$
$$\frac{dv}{dt} + \frac{v}{CR} = 0$$

Hence,

$$\boxed{v(t) = V_o e^{-\frac{t}{RC}} = V_o e^{-\frac{t}{\tau}},}$$

If $t = \tau$,

$$v(\tau) = V_o e^{-\frac{\tau}{t}} = V_o e^{-1}$$
$$= \frac{V_o}{e} = \frac{V_o}{2.718}$$
$$= 0.368 \times V_o$$

where $\tau = RC$.

This shows that the voltage response of RC circuit is an exponential decay of the initial voltage.

- Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the natural response of the circuit.

Remarks:

(a) $\tau = RC$ is the time constant which is the time required for the response to decay by 36.8 percent of its initial value.
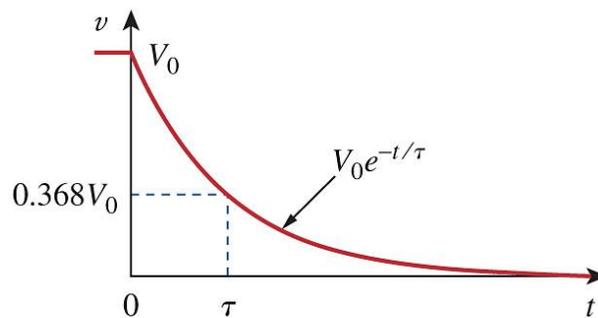
(b) The smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.
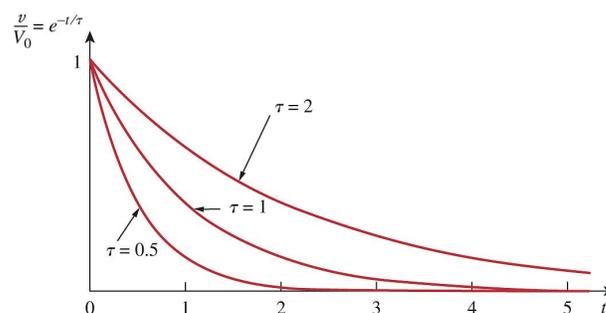
(c) The voltage v(t) is less than 1 percent of V_0 after 5τ (five time constants). Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.

(d) $i_R(t) = \frac{v(t)}{R} = \frac{V_o}{R}e^{\frac{-t}{\tau}}$.
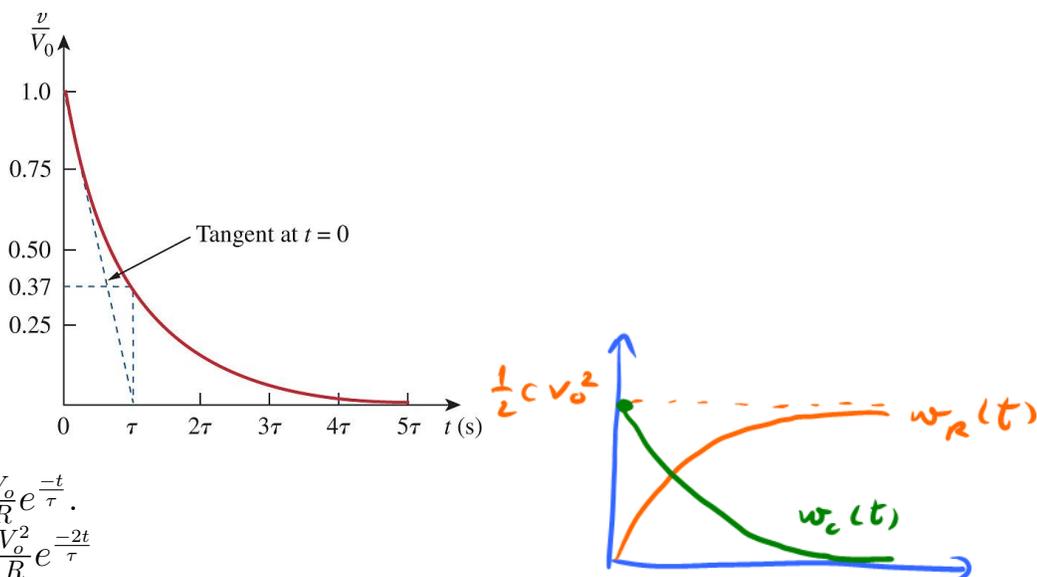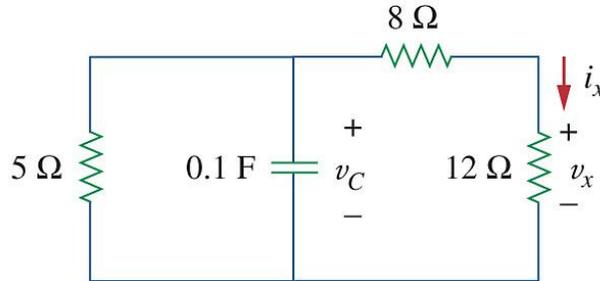
(e) $p_R(t) = vi_R = \frac{V_o^2}{R}e^{\frac{-2t}{\tau}}$

(f) $w_R(t) = \int_0^t p(\mu)d\mu = \frac{1}{2}CV_o^2(1 - e^{\frac{-2t}{\tau}}) = w_C(0) - w_C(t)$

   • Notice that as $t \to \infty$, $w_R \to \frac{1}{2}CV_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor.
   • The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

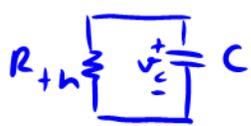In summary: The key in working with a source-free RC circuit is to determine:
1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant $\tau \ (= RC)$
   • For more complicated questions, it might not be obvious what value of R should be used. In which case, the R is the Thevenin equivalent resistance at the terminal of the capacitor (i.e., we take out the capacitor C and find $R = R_{TH}$ at its terminals)

Ex. Let $v_C(0) = 15$ V. Determine $v_C(t)$, $v_x(t)$, and $i_x(t)$ for $t > 0$.

$$i_x(t) = \frac{v_x(t)}{12\Omega}$$

$$= \left(\frac{9}{12}\right) \times e^{-2.5t} \text{ A}$$

$$\frac{9}{12} \to \frac{3}{4} \approx 0.75$$

$$R_{th} = (8+12)//5 \qquad \tau = R_{th} \times C \qquad v_c(t) = v(0) \times e^{-t/\tau}$$

$$= 20//5 \qquad = 4 \times 0.1 \qquad = 15 \times e^{-2.5t} \text{ V}$$

$$= 4\,\Omega \qquad = 0.4 \text{ sec} \qquad v_x(t) = \frac{12}{12+8} \times v_c(t) = \frac{3}{5} \times 15 \times e^{-2.5t}$$

$$\qquad = \frac{4}{10} = \frac{2}{5} \text{ sec.} \qquad = 9 e^{-2.5t} \text{ V}$$

Remark: The answer is the same if we replace $t > 0$ in the above question by $t \geq 0$.

Ex. The switch in the circuit below has been closed for a long time, and it is opened at $t = 0$. Find v(t) for $t \geq 0$. Calculate the initial energy stored in the capacitor.

$$w_c(0) = \frac{1}{2} \times C \times v_c^2(0)$$

$$= \frac{1}{2} \times (20 \times 10^{-3}) \times 15^2$$
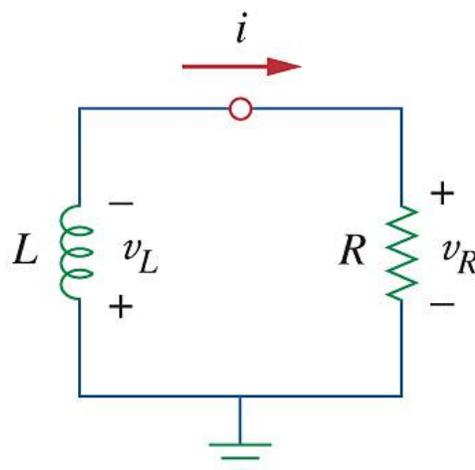
$$= 2.25 \text{ J}$$

For $t > 0$
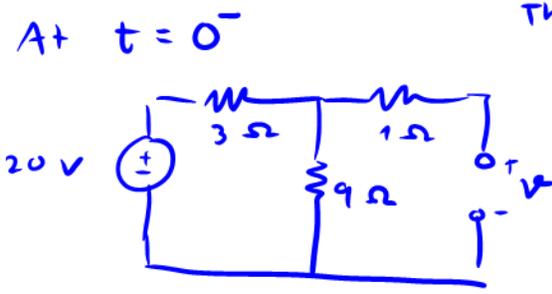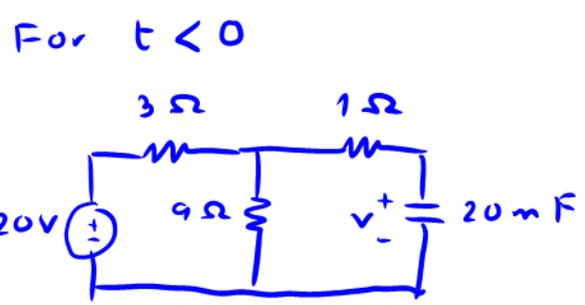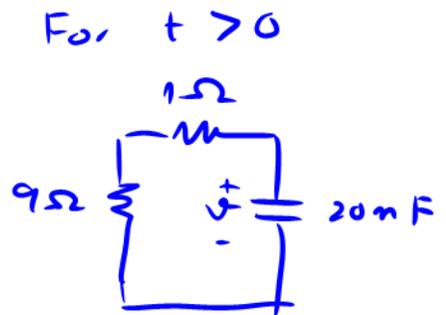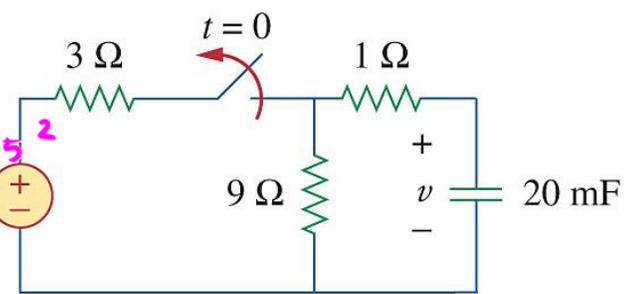
There can not be jump in v_c.

$$v_c(0) = 15 \text{ V}$$

$$\tau = R \times C = 10 \times 20mF$$

$$= 0.2 \text{ sec.}$$

$$\boxed{v(t) = 15 e^{-5t} \text{ V}} \qquad t \geq 0$$

For $t < 0$

At $t = 0^-$

$$v = 20 \times \frac{9}{3+9} = \frac{20 \times 9}{12} = 15 \text{ V}$$

7.2. Source-Free RL Circuits

Consider the series connection of a resistor and an inductor.   Assume

$$i_L(t) = i_L(t_o) e^{-(t-t_o)/\tau}$$

$$\tau = \frac{L}{R}$$

that the inductor has an initial current I_o or $i(0) = I_o$.(hence, the energy stored in the inductor is $w(0) = \frac{1}{2}LI_o^2$.)

Applying KVL, we get

$$v_L + v_R = 0$$
$$L\frac{di}{dt} + Ri = 0$$
$$\frac{di}{dt} + \frac{R}{L}i = 0$$

Hence,

$$i(t) = I_o e^{\frac{-Rt}{L}} = I_o e^{\frac{-t}{\tau}}.$$

Note that:

(a) $\tau = \frac{L}{R}$ is the time constant which is the time required for the response to decay by 36.8 percent of its initial value.

(b) $v_R(t) = iR = I_o R e^{\frac{-t}{\tau}}$.

(c) $p(t) = v_R i = I_o^2 R e^{\frac{-2t}{\tau}}$.

(d) $w_R(t) = \int_0^t p(\mu)d\mu = \frac{1}{2}LI_o^2(1 - e^{\frac{-2t}{\tau}}) = w_L(0) - w_L(t)$.

In summary: The key in working with a source free RL circuit is to determine:
(a) The initial current $i(0) = I_o$.
(b) The time constant $\tau \ (= \frac{L}{R})$.
      • In general, R is the Thevenin equivalent resistance at the terminal of the inductor.(We take out the inductor L and find $R = R_{TH}$ at its terminals.)

Ex. The switch in the circuit below has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.
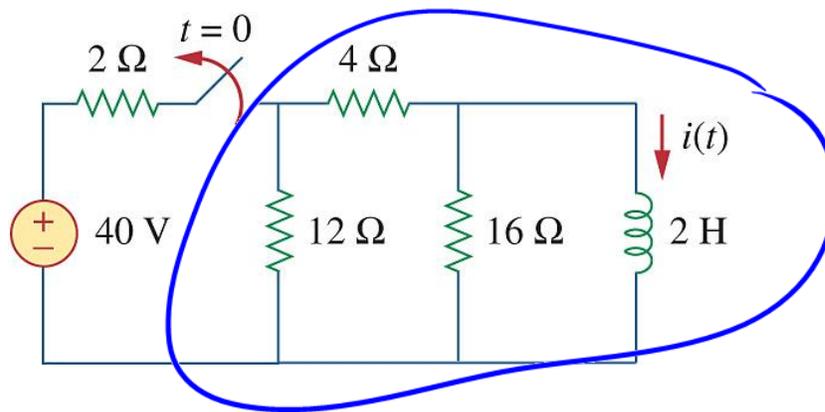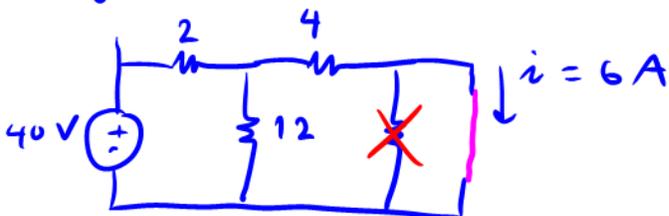

$t < 0$
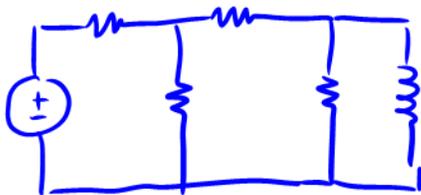
$t > 0$

$$R_{th} = 8 \, \Omega$$

$$\tau = \frac{L}{R} = \frac{2H}{8\Omega} = \frac{1}{4} \ s$$

$$i(t) = i(0) \, e^{-t/\tau}$$

$$= 6 e^{-4t} \ A .$$

$t = 0^-$

$i = 6 A$

$i(0^-) = 6 \ A$

$i(0) = i(0^-) = 6A$

Ex.7.5 Determine i, i_o and v_o for all t in the circuit. Assume that the switch was closed for a long time.