

ECS 203 (CPE2)

Asst. Prof. Dr.Prapun Suksompong

CHAPTER 10

First-Order Circuits

We have been assuming that the circuits under our consideration have reached their steady-states. This assumption has helped us analyze circuits under dc conditions in Chapter 6 (see Examples 6.2.11, 6.4.13, and 6.4.14) and perform ac analysis in Chapters 7, 8, and 9.

In this chapter and the next chapter, we relax the steady-state assumption. This means we will need to return to solving differential equations. We will start with circuits that contain only one (equivalent) capacitor or inductor. This is enough to see how the voltage or current behaves during the charging/discharging of these storage elements.

10.1. Introduction and a Mathematical Fact

10.1.1. We will examine two types of simple circuits with “one” storage element:

- (a) A circuit with a resistor and one capacitor (called an **RC circuit**);
and
- (b) A circuit with a resistor and an inductor (called an **RL circuit**).

These circuits may look simple but they find applications in electronics, communications, and control systems. Our focus will be on the RC circuit.

10.1.2. Applying Kirchhoff’s laws to the RC and RL circuits produce **first order differential equations**. Hence, the circuits are collectively known as **first-order circuits**.

10.1.3. There are two ways to excite the circuits.

- (a) By initial conditions of the storage elements in the circuit.
 - Also known as *source-free circuits*
 - Assume that energy is initially stored in the capacitive or inductive element.
 - This is the **discharging process**.
- (b) By using independent sources
 - This is the **charging process**
 - For this chapter, we will consider independent dc sources.

Before we start our circuit analysis, it is helpful to consider one mathematical fact which we will use throughout this chapter:

10.1.4. The solution of the first-order differential equation

$$\frac{d}{dt}x(t) = ax + b$$

is given by

$$(10.11) \quad x(t) = e^{a(t-t_0)} \left(x(t_0) + \frac{b}{a} \right) - \frac{b}{a}.$$

This result takes a bit of effort to proof. However, if you have MATLAB, you can get 10.11 using one line of code:

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dsolve('Dx = a*x+b', 'x(t0) = x0', 't')
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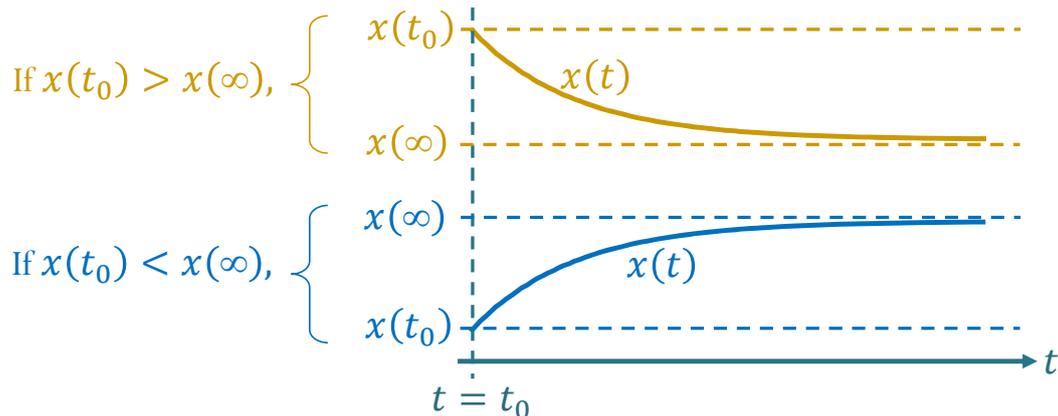
In addition, if $a < 0$, the solution is given by

$$(10.12) \quad \boxed{x(t) = e^{a(t-t_0)} (x(t_0) - x(\infty)) + x(\infty).}$$

where $x(\infty) = \lim_{t \rightarrow \infty} x(t)$.

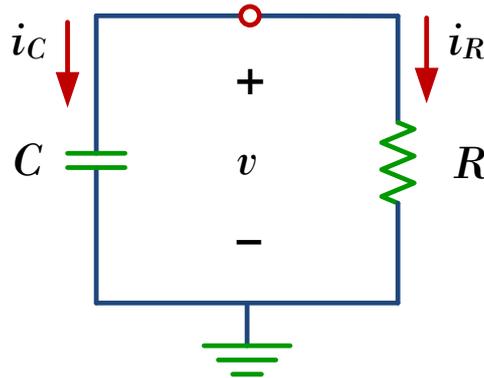
When $a < 0$, from (10.11), $x(\infty) = -b/a$. We can then get (10.12) by substituting b/a in (10.11) with $x(\infty)$.

10.1.5. It turns out that you only have two kinds of plots to worry about when dealing with (10.12):



10.2. Source-Free RC Circuits

DEFINITION 10.2.1. A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistor.



10.2.2. Consider a series combination of a resistor and an initially charged capacitor. We assume that at time $t = 0$, the initial voltage is $v(0) = V_0$. (Hence, the initial stored energy is $w(0) = \frac{1}{2}CV_0^2$.)

Applying KCL, we get

$$\begin{aligned} i_C + i_R &= 0 \\ C \frac{dv}{dt} + \frac{v}{R} &= 0 \\ \frac{dv}{dt} &= \left(-\frac{1}{RC} \right) v \end{aligned}$$

Hence,

$$(10.13) \quad \boxed{v(t) = V_0 e^{-\frac{t}{RC}} = V_0 e^{-\frac{t}{\tau}},}$$

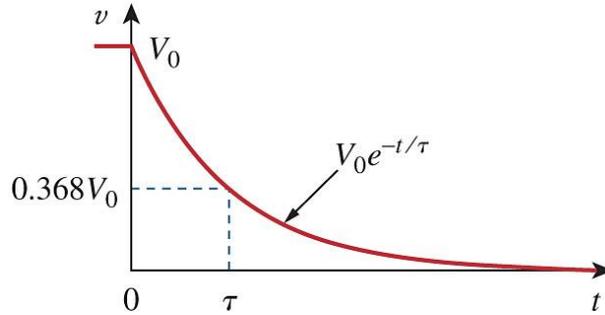
where $\tau = RC$.

This shows that the voltage response of RC circuit is an exponential decay of the initial voltage.

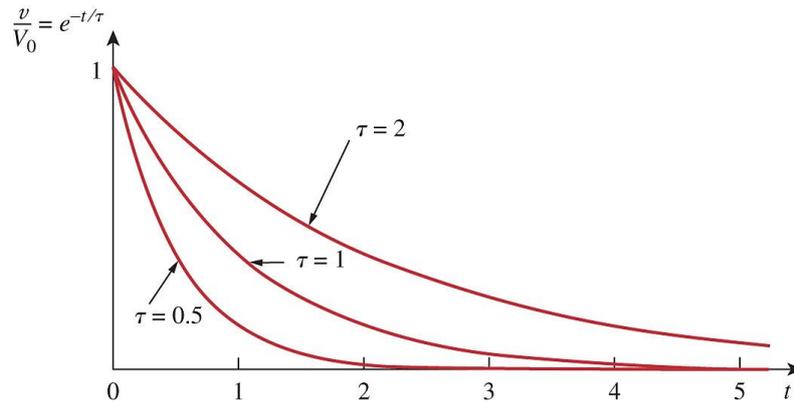
- Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

10.2.3. Remarks:

- (a) $\tau = RC$ is the time constant which is the time required for the response to decay to 36.8 percent of its initial value. ($e \approx 2.718$ and hence $1/e \approx 0.368$.)



- (b) The smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.



(c) Energy Consideration:

(i) Capacitor:

$$w_C(t) = \frac{1}{2} C v^2(t) = \frac{1}{2} C \left(V_0 e^{-\frac{t}{\tau}} \right)^2 = \frac{1}{2} C V_0^2 e^{-\frac{2t}{\tau}} = w_C(0) e^{-\frac{2t}{\tau}}$$

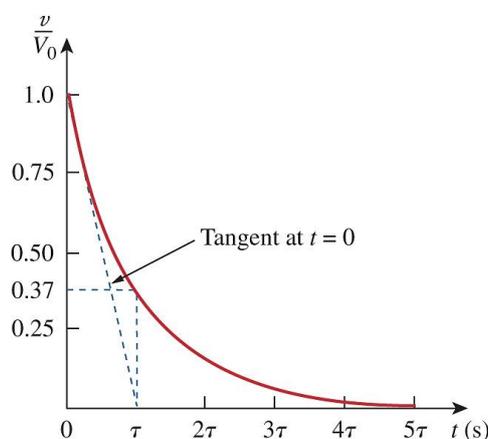
(ii) Resistor:

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}}$$

$$p_R(t) = v(t) \times i_R(t) = \frac{V_0^2}{R} e^{-\frac{2t}{\tau}}$$

$$w_R(t) = \int_0^t p(\mu) d\mu = \frac{1}{2} C V_0^2 (1 - e^{-\frac{2t}{\tau}}) = w_C(0) - w_C(t)$$

- Notice that as $t \rightarrow \infty$, $w_R \rightarrow \frac{1}{2}CV_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor.
 - The energy that was initially stored in the capacitor is eventually dissipated in the resistor.
- (d) The voltage $v(t)$ is less than 1 percent of V_0 after 5τ (five time constants). Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.



10.2.4. In summary: The key in working with a source-free RC circuit is to determine the voltage across the capacitor which can be found from :

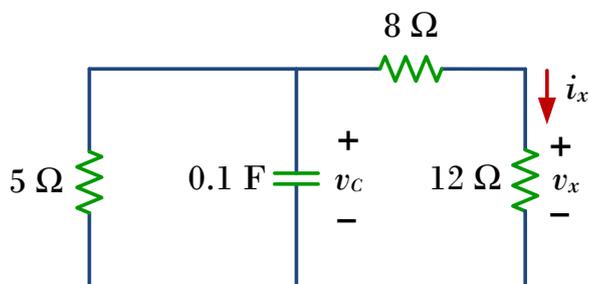
1. the initial voltage $v(0) = V_0$.
2. the time constant $\tau (= RC)$
3. Equation (10.13)

10.2.5. There are basically three extensions of what we found:

- (a) The initial condition $v(t_0)$ can be given at non-zero t_0 . This has already been taken care of by (10.12).

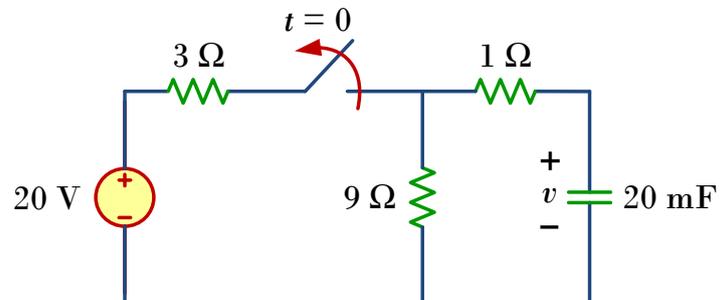
- (b) There can be more than one resistors. In which case, find the equivalent resistance at the capacitor terminals. See Example 10.2.6.
- (c) The initial value of the voltage $v(t_0)$ might not be given but condition(s) of the circuit before time t_0 is given instead. See Example 10.2.8. In which case, we may have to consider the long-term (steady-state) behavior of the circuit before t_0 . In such situation, recall two properties of capacitors studied in 6.2.7:
- (i) For DC, capacitor becomes open circuit.
 - (ii) Voltage across the capacitor cannot change instantaneously.

EXAMPLE 10.2.6. Let $v_C(0) = 15$ V. Determine $v_C(t)$, $v_x(t)$, and $i_x(t)$ for $t \geq 0$.



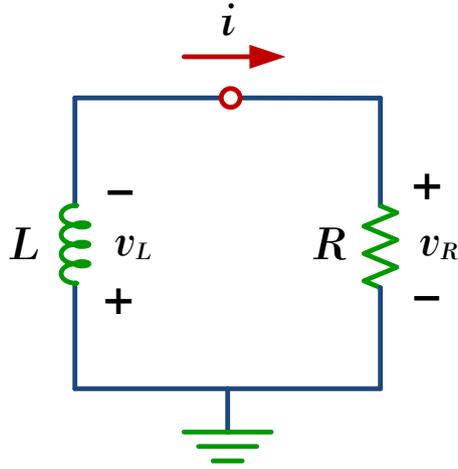
EXAMPLE 10.2.7. Circuit with switch(es):

EXAMPLE 10.2.8. The switch in the circuit below has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



10.3. Source-Free RL Circuits

10.3.1. Consider the series connection of a resistor and an inductor.



Assume that, at time $t = t_0$, the inductor has an initial current $i(t_0) = I_0$. (Hence, the energy stored in the inductor is $w(0) = \frac{1}{2}LI_0^2$.)

Applying KVL, we get

$$\begin{aligned} v_L + v_R &= 0 \\ L \frac{di}{dt} + Ri &= 0 \\ \frac{di}{dt} &= \left(-\frac{R}{L} \right) i \end{aligned}$$

From (10.12),

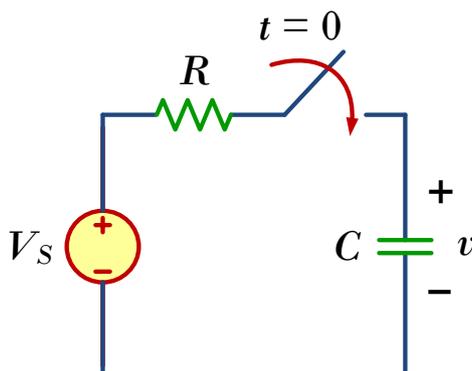
$$i(t) = I_0 e^{-\frac{t-t_0}{\tau}} = I_0 e^{\frac{-R}{L}(t-t_0)},$$

where $\tau = \frac{L}{R}$ is the **time constant** which, again, is the time required for the response to decay to 36.8 percent of its initial value.

10.4. Step Response of an RC Circuit

In the previous sections, we have been considering RC and RL circuits with no source. Here, we start to look at the cases with dc sources.

10.4.1. Consider an RC circuit with voltage step input below:



The voltage across the capacitor is v . We assume an initial voltage¹ $v(0) = V_0$ on the capacitor.

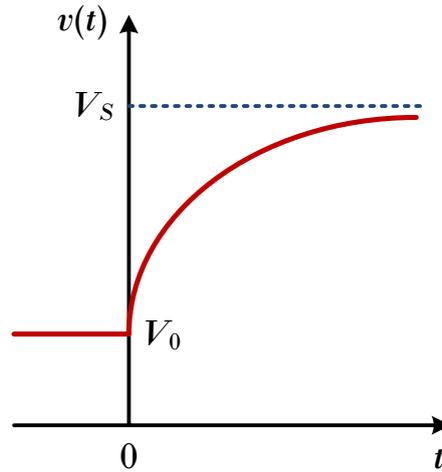
For $t \geq 0$, applying KCL, we have

$$\begin{aligned} C \frac{dv}{dt} + \frac{v - V_s}{R} &= 0 \\ \frac{dv}{dt} + \frac{v}{RC} &= \frac{V_s}{RC} \\ \frac{dv}{dt} &= \left(-\frac{1}{RC} \right) v + \frac{V_s}{RC} \end{aligned}$$

From (10.11), the solution is

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t \geq 0.$$

¹Since the voltage of a capacitor cannot change instantaneously $v(0^-) = v(0^+) = V_0$, where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching.



10.4.2. In general, the step response of an RC circuit whose switch changes position at time $t = 0$ can be written as

$$(10.14) \quad \boxed{v(t) = v(\infty) + (v(0) - v(\infty))e^{-\frac{t}{\tau}}, \quad t > 0.}$$

where $v(0)$ is the initial voltage and $v(\infty)$ is the final or steady-state value. We obtain

- (a) $v(0)$ from the given circuit for $t < 0$ and
- (b) $v(\infty)$ and τ from the circuit for $t > 0$.

10.4.3. If the switch changes position at time $t = t_o$ instead of at $t = 0$, the response becomes

$$(10.15) \quad v(t) = v(\infty) + (v(t_o) - v(\infty))e^{-\frac{(t-t_o)}{\tau}}, \quad t > t_o$$

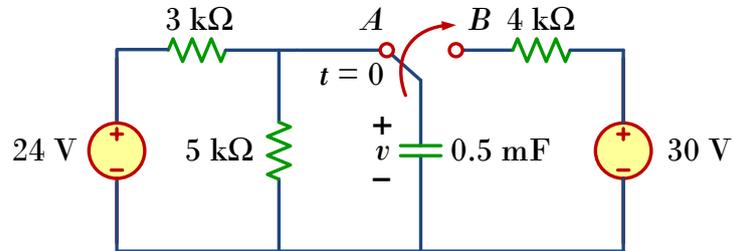
where $v(t_o)$ is the initial value at $t = t_o$.

10.4.4. The key in finding the step response of an RC circuit is to determine the voltage across the capacitor which can be found from:

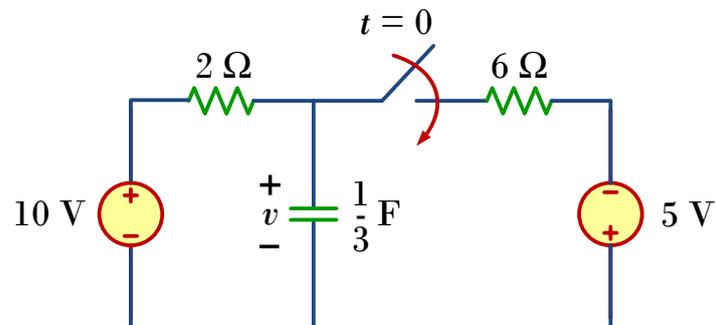
1. the initial capacitor voltage $v(0)$ or $v(t_o)$,
2. the final capacitor voltage $v(\infty)$,
3. the time constant τ ,
4. Equations (10.14) or (10.15).

10.4.5. Extension:

EXAMPLE 10.4.6. The switch in the circuit below has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



EXAMPLE 10.4.7. Find $v(t)$ for $t > 0$ in the circuit below. Assume the switch has been open for a long time and is closed at $t = 0$. Numerically evaluate $v(t)$ at $t = 0.5$.



10.4.8. Steps for solving RC Circuit with SW(s)

Step 0 : Partition the time axis into multiple segments where each cut happens at the time when a SW changes its configuration.

Work with the time intervals chronologically.

Draw the circuit corresponding to each SW configuration (during each time interval).

Focus on the voltage $v(t)$ across the capacitor first.

Step 1 : Let t_0 be the time of the first cut. (The first time that there is a change in the SW configuration.)

1.1 In many cases, $v(t_0)$ is given, say $v(t_0) = V_0$.

1.2 If not, consider the circuit over the time interval $(-\infty, t_0)$.

(i) Because the circuit has been in this state for a long time (from $t = -\infty$), we know that, at any time $t < t_0$, the circuit has already reached its steady state (dc conditions).

(ii) Find $v(t)$ for $t < t_0$.

Note that because this is in dc conditions, $v(t)$ is just a number, say, V_0 .

(iii) In particular, we know that $v(t_0^-) = V_0$.

(iv) Because there can't be any jump (sudden change) in the voltage value across the capacitor,

$$v(t_0) = v(t_0^-) = V_0.$$

Step 2 Consider the circuit during the next time interval. (This will be $[t_0, \infty)$ if there is only one SW and it only changes its position one time. Otherwise, this will be an interval of the form $[t_0, t_1)$.)

(i) Reduce the non-capacitor part into its Thevenin equivalent.

Now the circuit should look like this:

Note that for the source-free case, $V_{th} = 0$.

(ii) Find the time constant $\tau = R_{th}C$.

(iii) For $t_0 \leq t < t_1$,

$$\begin{aligned}v(t) &= e^{-\frac{t-t_0}{\tau}} (v(t_0) - v(\infty)) + v(\infty) \\ &= e^{-\frac{t-t_0}{\tau}} (V_0 - V_{th}) + V_{th}.\end{aligned}$$

(iv) To find the values of other voltage or current, go back to the circuit before finding the Thevenin equivalent circuit. Treat the voltage $v(t)$ across the capacitor solved in the previous step as a voltage source.

Step 3 If there are still more time interval(s), repeat step 2 for the next time interval. The end point of the time interval from the previous step becomes the “initial time” of this interval. The voltage at the time just before the new initial time can be found from the expression in the previous step.