CHAPTER 1

Basic Concepts

In electrical engineering, we are often interested in communicating or transferring energy from one point to another. To do this requires an interconnection of electrical devices. Such interconnection is referred to as an electric circuit, and each component of the circuit is known as an element.

Definition 1.0.1. An electric circuit is an interconnection of electrical elements.

- Examples of circuit elements are discussed in Section 1.3.

1.1. International System of Units

1.1.1. As engineers, we deal with measurable quantities. Our measurement must be communicated in standard language that virtually all professionals can understand irrespective of the country. Such an international measurement language is the International System of Units (SI).
1.1.2. In this system, there are six principal units from which the units of all other physical quantities can be derived.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Basic Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>meter</td>
<td>m</td>
</tr>
<tr>
<td>Mass</td>
<td>kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>Time</td>
<td>second</td>
<td>s</td>
</tr>
<tr>
<td>Electric Current</td>
<td>ampere</td>
<td>A</td>
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<tr>
<td>Temperature</td>
<td>kelvin</td>
<td>K</td>
</tr>
<tr>
<td>Luminous Intensity</td>
<td>candela</td>
<td>cd</td>
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</tbody>
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1.1.3. One great advantage of SI unit is that it uses prefixes based on the power of 10 to relate larger and smaller units to the basic unit.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Prefix</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{12}$</td>
<td>tera</td>
<td>T</td>
</tr>
<tr>
<td>$10^9$</td>
<td>giga</td>
<td>G</td>
</tr>
<tr>
<td>$10^6$</td>
<td>mega</td>
<td>M</td>
</tr>
<tr>
<td>$10^3$</td>
<td>kilo</td>
<td>k</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>centi</td>
<td>c</td>
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<tr>
<td>$10^{-3}$</td>
<td>milli</td>
<td>m</td>
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<tr>
<td>$10^{-6}$</td>
<td>micro</td>
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<tr>
<td>$10^{-9}$</td>
<td>nano</td>
<td>n</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>pico</td>
<td>p</td>
</tr>
</tbody>
</table>

Example 1.1.4. Change of units:

$$600,000,000 \text{ mA} = 6 \times 10^9 \times 10^{-3} \text{ A} = 6 \times 10^6 \text{ A} = 6 \times 10^6 \frac{\text{A}}{10} = 600 \text{ kA}$$

1.2. Circuit Variables

1.2.1. **Charge:** The concept of electric charge is the underlying principle for all electrical phenomena. Charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C). The charge of an electron is $-1.602 \times 10^{-19}$ C.

- The coulomb is a large unit for charges. In 1 C of charge, there are $1/(1.602 \times 10^{-19}) = 6.24 \times 10^{18}$ electrons. Thus realistic or laboratory values of charges are on the order of pC, nC, or µC.
- A large power supply capacitor can store up to 0.5 C of charge.
1.2. Law of Conservation of Charge: Charge can neither be created nor destroyed, only transferred.

Definition 1.2.3. **Current**: The time rate of change of charge, measured in amperes (A). Mathematically,

\[ i(t) = \frac{d}{dt} q(t) \]

Note:
- 1 ampere (A) = 1 coulomb/second (C/s).
- The charge transferred between time \( t_1 \) and \( t_2 \) is obtained by

\[ \Delta q = \int_{t_1}^{t_2} i(t) \, dt \]

1.2.4. Representing current in circuit (schematic) diagram:
- To talk about current, we need to specify its direction and amount.
- These are conveyed by the direction of the arrow (referred to as the reference direction) and the labeled value (which can be negative if the current actually flows in the opposite direction.)

1.2.5. Two types of currents:
(a) A **direct current** (dc) is a current that remains constant with time.

- Produced by sources such as batteries, power supplies.
(b) A **time-varying current** is a current that **varies with time**.

- An **alternating current** (ac) is a type of time-varying current that varies sinusoidally with time.

1.2.6. By convention the symbol $I$ is used to represent such a constant current. A time-varying current is represented by the symbol $i$.

**Definition 1.2.7. Voltage** (or potential difference) across an element: the energy required to move a unit charge through an element, measured in volts (V).

- $1$ volt (V) = $1$ joule/coulomb = $1$ newton-meter/coulomb

- Representing voltage in circuit (schematic) diagram: To talk about voltage, we need to specify
  (a) its polarity via the plus (+) and minus (-) symbols at the two positions (points $a$ and $b$ in the picture above)
  (b) its value (can be a negative number)
• The voltage between two points \( a \) and \( b \) in a circuit is denoted by \( v_{ab} \) and can be interpreted in two ways:

\[
\begin{align*}
\text{(a) point } a & \text{ is at a potential of } v_{ab} \text{ volts higher than point } b, \text{ or} \\
\text{(b) the potential at point } a & \text{ with respect to point } b \text{ is } v_{ab}.
\end{align*}
\]

\( v_{ab} = -v_{ba} \)

• Mathematically,

\[
v_{ab} = \frac{dw}{dq}
\]

where \( w \) is the energy in joules (J) and \( q \) is charge in coulombs (C).

1.2.8. Like electric current, a constant voltage is called a **dc voltage** and is represented by \( V \), whereas a **time-varying** voltage is represented by \( v \). A time-varying voltage that is sinusoidal is called an **ac voltage**.

**Example 1.2.9.** A dc voltage is commonly produced by a battery; ac voltage is produced by an electric generator.

1.2.10. Current and voltage are the two basic variables in electric circuits. The common term **signal** is used for an electric quantity such as a current or a voltage (or even electromagnetic wave) when it is used for conveying information. Engineers prefer to call such variables **signals** rather than mathematical functions of time because of their importance in communications and other disciplines.

For practical purposes, we need to be able to find/calculate/measure more than the current and voltage. We all know from experience that a 100-watt bulb gives more light than a 60-watt bulb. We also know that when we pay our bills to the electric utility companies, we are paying for the electric energy consumed over a certain period of time. Thus power and energy calculations are important in circuit analysis.

**Definition 1.2.11.** **Power:** time rate of **absorbing** (or **expending**) energy, measured in watts (W). Mathematically, the instantaneous power

\[
p = \frac{dw}{dt} = \frac{dw}{dq} \frac{dq}{dt} = vi
\]
**Definition 1.2.12.** Passive Sign Convention (PSC): Sign of power

(a) **Plus sign:** Power is **absorbed** (consumed) by the element. (resistor)
   - For resistor, we say the power is dissipated.

(b) **Minus sign:** Power is **supplied** by the element. (battery, generator)

1.2.13. To comply with the convention, we have to be careful when applying formula (1.1).

(a) If the current “enters” the element through the “positive terminal” of the voltage across this element, \( p = vi \).

\[
\begin{align*}
\text{To find} & \quad \text{the amount of power absorbed by an element,} \\
\text{“enter +”} & \quad i \quad \text{“enter -”} \\
\end{align*}
\]

\[
\begin{align*}
\text{p} & = +vi \\
\text{p} & = -vi
\end{align*}
\]

(b) If the current “enters” the element through the “negative terminal” of the voltage across this element, \( p = -vi \).

Note that, here, to choose which version of the formula to apply, we only consider the direction of the current’s arrow and the polarity of the voltage. We do not have to care whether the variables \( i \) and \( v \) are positive or negative.

**Example 1.2.14.**

\[
\begin{align*}
\text{3 A} & \quad \text{“enter +”} \\
\text{“enter -”} & \quad \text{3 A} \\
\end{align*}
\]

\[
\begin{align*}
p & = +1V \\
& = 3 \times 4 \\
& = 12 \text{ W}
\end{align*}
\]

\[
\begin{align*}
p & > 0 \Rightarrow \text{absorbing power}
\end{align*}
\]
Example 1.2.15. Light bulb or battery: which is which?

1.2.16. Law of Conservation of Energy: Energy can neither be created nor destroyed, only transferred.

- For this reason, the algebraic sum of power in a circuit, at any instant of time, must be zero.
- The total power supplied to the circuit must balance the total power absorbed.

Example 1.2.17. The circuit below has five elements. Let $P_k$ be the power “absorbed” by element $k$. (Although we use the word “absorbed”, if $P_k$ is negative, then element $k$ is actually supplying power.)

If $P_1 = -205 \, \text{W}$, $P_2 = 60 \, \text{W}$, $P_4 = 45 \, \text{W}$, $P_5 = 30 \, \text{W}$, calculate the power $P_3$. 

$$-205 + 60 + P_3 + 45 + 30 = 0$$

$$P_3 = 70 \, \text{W} > 0$$

absorbing power
1.2.18. **Energy**: the energy absorbed or supplied by an element from time $t = t_1$ to $t = t_2$ is

$$w = \int_{t_1}^{t_2} p(t) \, dt = \int_{t_1}^{t_2} v(t)i(t) \, dt.$$  

- Integration suggests finding area under the curve.
- Need to be careful with negative area.

**Example 1.2.19. Electricity bills**: The electric power utility companies measure energy in kilowatt-hours (kWh), where $1 \text{ kWh} = 3600 \text{ kJ}$.

### 1.3. Circuit Elements

**Definition 1.3.1.** There are 2 types of elements found in electrical circuits.

1) **Active elements** (is capable of generating energy), e.g., generators, batteries, and operational amplifiers (Op-amp).

2) **Passive element**, e.g., resistors, capacitors and inductors.

**Definition 1.3.2.** The most important active elements are voltage and current sources:

(a) **Voltage source** provides the circuit with a specified voltage (e.g. a 1.5V battery)

(b) **Current source** provides the circuit with a specified current (e.g. a 1A current source).

**Definition 1.3.3.** In addition, we may characterize the voltage or current sources as:

1) **Independent source**: An active element that provides a specified voltage or current that is completely independent of other circuit elements.
2) **Dependent source**: An active element in which the source quantity is controlled by another voltage or current.

1.3.4. The key idea to keep in mind is that a voltage source comes with polarities (+ -) in its symbol, while a current source comes with an arrow, irrespective of what it depends on.

**Example 1.3.5.** Current-controlled voltage source

**Example 1.3.6.** Draw the general form of a voltage-controlled current source.
Example 1.3.7. Indicate the type of the dependent source in each of the circuit in the figure below.

1.3.8. Remarks:
- Dependent sources are useful in modeling elements such as transistors, operational amplifiers and integrated circuits.
- Ideal sources
  - An ideal voltage source (dependent or independent) will produce any current required to ensure that the terminal voltage is as stated.
  - An ideal current source will produce the necessary voltage to ensure the stated current flow.
  - Thus an ideal source could in theory supply an infinite amount of energy.
- Not only do sources supply power to a circuit, they can absorb power from a circuit too.
- For a voltage source, we know the voltage but not the current supplied or drawn by it. By the same token, we know the current supplied by a current source but not the voltage across it.
Here we explore two fundamental laws that govern electric circuits (Ohm’s law and Kirchhoff’s laws) and discuss some techniques commonly applied in circuit design and analysis.

2.1. Ohm’s Law

Ohm’s law shows a relationship between voltage and current of a resistive element such as conducting wire or light bulb.

2.1.1. Ohm’s Law: The voltage $v$ across a resistor is directly proportional to the current $i$ flowing through the resistor.

$$v = iR,$$

where $R =$ resistance of the resistor, denoting its ability to resist the flow of electric current. The resistance is measured in ohms ($\Omega$).

- To apply Ohm’s law, the direction of current $i$ and the polarity of voltage $v$ must conform with the passive sign convention. This implies that current flows from a higher potential to a lower potential.
in order for \( v = iR \). If current flows from a lower potential to a higher potential, \( v = -iR \).

2.1.2. The resistance \( R \) of a cylindrical conductor of cross-sectional area \( A \), length \( L \), and conductivity \( \sigma \) is given by

\[
R = \frac{L}{\sigma A}
\]

Alternatively,

\[
R = \frac{\rho L}{A}
\]

where \( \rho \) is known as the resistivity of the material in ohm-meters. Good conductors, such as copper and aluminum, have low resistivities, while insulators, such as mica and paper, have high resistivities.

2.1.3. Remarks:
(a) \( R = v/i \)
(b) Conductance :

\[
G = \frac{1}{R} = \frac{i}{v}
\]

The unit of \( G \) is the mho \( (\Omega) \) or siemens \( (S) \)

\[\text{1Yes, this is NOT a typo! It was derived from spelling ohm backwards.}\]
\[\text{2In English, the term siemens is used both for the singular and plural.}\]
(c) The two extreme possible values of $R$.

(i) When $R = 0$, we have a short circuit and

$$v = iR = 0$$

showing that $v = 0$ for any $i$. 

(ii) When $R = \infty$, we have an open circuit and

$$i = \lim_{R \to \infty} \frac{v}{R} = 0$$

indicating that $i = 0$ for any $v$.

2.1.4. A resistor is either fixed or variable. Most resistors are of the fixed type, meaning their resistance remains constant.
A common variable resistor is known as a potentiometer or pot for short.

2.1.5. Not all resistors obey Ohm’s law. A resistor that obeys Ohm’s law is known as a linear resistor.

- A nonlinear resistor does not obey Ohm’s law.

- Examples of devices with nonlinear resistance are the lightbulb and the diode.

- Although all practical resistors may exhibit nonlinear behavior under certain conditions, we will assume in this class that all elements actually designated as resistors are linear.
2.1. Using Ohm’s law, the power \( p \) dissipated by a resistor \( R \) is
\[
p = vi = i^2 R = \frac{v^2}{R}.
\]

**Example 2.1.7.** In the circuit below, calculate the current \( i \), and the power \( p \).

```
30 V + 5 kΩ -
```

**Definition 2.1.8.** The **power rating** is the maximum allowable power dissipation in the resistor. Exceeding this power rating leads to overheating and can cause the resistor to burn up.

**Example 2.1.9.** Determine the minimum resistor size that can be connected to a 1.5V battery without exceeding the resistor's \( \frac{1}{4} \)-W power rating.

\[
\frac{(1.5)^2}{R} = \frac{v^2}{R} = p \leq \frac{1}{4}
\]

\[
\frac{3 \times \frac{3}{2} \times 4}{2} = (1.5)^2 \times 4 \leq R
\]

\[
R \geq 9 \, \Omega
\]

So, \( \text{min } R = 9 \, \Omega \).
2.2. Node, Branches and Loops

DEFINITION 2.2.1. Since the elements of an electric circuit can be interconnected in several ways, we need to understand some basic concept of network topology.

• Network = interconnection of elements or devices
• Circuit = a network with closed paths

DEFINITION 2.2.2. Branch: A branch represents a single element such as a voltage source or a resistor. A branch represents any two-terminal element.

DEFINITION 2.2.3. Node: A node is the “point” of connection between two or more branches.

• It is usually indicated by a dot in a circuit.
• If a short circuit (a connecting wire) connects two nodes, the two nodes constitute a single node.

DEFINITION 2.2.4. Loop: A loop is any closed path in a circuit. A closed path is formed by starting at a node, passing through a set of nodes and returning to the starting node without passing through any node more than once.

Definition 2.2.5. Series: Two or more elements are in series if they are cascaded or connected sequentially and consequently carry the same current.

Definition 2.2.6. Parallel: Two or more elements are in parallel if they are connected to the same two nodes and consequently have the same voltage across them.
2.2.7. Elements may be connected in a way that they are neither in series nor in parallel.

Example 2.2.8.

Example 2.2.9. How many branches and nodes does the circuit in the following figure have? Identify the elements that are in series and in parallel.

2.2.10. A loop is said to be independent if it contains a branch which is not in any other loop. Independent loops or paths result in independent sets of equations. A network with \( b \) branches, \( n \) nodes, and \( \ell \) independent loops will satisfy the fundamental theorem of network topology:

\[
\ell = b - n + 1
\]

Definition 2.2.11. The primary signals within a circuit are its currents and voltages, which we denote by the symbols \( i \) and \( v \), respectively. We define a branch current as the current along a branch of the circuit, and a branch voltage as the potential difference measured across a branch.
2.3. Kirchhoff’s Laws

Ohm’s law coupled with Kirchhoff’s two laws gives a sufficient, powerful set of tools for analyzing a large variety of electric circuits.

2.3.1. **Kirchhoff’s current law (KCL):** the algebraic sum of currents departing a node (or a closed boundary) is zero. Mathematically,

\[ \sum_{n} i_n = 0 \]

KCL is based on the law of conservation of charge. An alternative form of KCL is

Sum of currents (or charges) drawn as entering a node

= Sum of the currents (charges) drawn as leaving the node.

**Example 2.3.2.**

\[ -\dot{i}_1 + \dot{i}_2 - \dot{i}_3 - \dot{i}_4 + \dot{i}_5 = 0 \]

**Example 2.3.3.** An important and frequently-used result:

If two or more elements are connected in series, they carry the same current. (defn. 2.2.5)

Apply KCL @ node a:

\[ (-I_1) + (I_2) = 0 \]

\[ I_1 = I_2 \]

Apply KCL @ node b:

\[ (-I_2) + I_3 = 0 \]

\[ I_2 = I_3 \]
Example 2.3.4. A simple application of KCL is combining current sources in parallel.

Note that KCL also applies to a closed boundary. This may be regarded as a generalized case, because a node may be regarded as a closed surface shrunk to a point. In two dimensions, a closed boundary is the same as a closed path. The total current entering the closed surface is equal to the total current leaving the surface.

Example 2.3.5.
A Kirchhoff’s voltage law (KVL): the algebraic sum of all voltages around a closed path (or loop) is zero. Mathematically,

\[ \sum_{m=1}^{M} v_m = 0 \]

Our convention is to go clockwise (cw).

KVL is based on the law of conservation of energy. An alternative form of KVL is

Sum of voltage drops = Sum of voltage rises.

**Example 2.3.6.**

**Example 2.3.7.** When voltage sources are connected in series, KVL can be applied to obtain the total voltage.
\[-10 + (-10) + 20 = 0\]

\[\text{\textcolor{red}{15} - (-10) - 15 = 0}\]

\[10 + 15 - 20 - 5 = 0\]

\[\nu_{ca} = \text{?} = 25 \text{V}\]

\[10 + 15 - \nu_{ca} = 0\]

\[\nu_{ca} = 25 \text{V}\]

\[10 + 15 - 10 - 15 = 0\]
Example 2.3.8. Find \( v_1 \) and \( v_2 \) in the following circuit.

![Circuit Diagram]

\[ v_1 = I \times 4 = 3 \times 4 = 12 \, \text{V} \]
\[ v_2 = -I \times 2 = -3 \times 2 = -6 \, \text{V} \]

Example 2.3.9. An important and frequently used result:

\[ v_1 - v_2 = 0 \quad v_1 = v_2 \]  
\[ v_2 - v_3 = 0 \quad v_2 = v_3 \]

Example 2.3.10 (HRW p. 725). In the figure below, all the resistors have a resistance of 4.0 \( \Omega \) and all the (ideal) batteries are 4.0 V. What is the current through resistor \( R \)?

\[ I = \frac{V}{R} = \frac{9}{4} = 2 \, \text{A} \]
Review: KVL and Ohm's law

\[ V_{BD} + 6 + 5 = 0 \]
\[ V_{BD} = -11 \]

Soon, we will be finding \( I_1 \), \( I_2 \), and \( I_3 \) on our own in Ex. 3.24. For now, suppose we get
\[ I_1 = I_4 = 1 \text{A} \] and
\[ I_2 = 0 \text{A}. \]

Find
\[ V_{AC} = 10 \text{V} \]
\[ V_{CA} = -V_{AC} = -10 \text{V} \]
\[ V_{BD} = -11 \text{V} \]

KVL:
\[ V_{AC} - 10 - 10 = 0 \]
\[ V_{AC} = 10 \text{V} \]

We use Ohm's law to find the voltage across each of the resistors:

\[ V_{DA} = I_1 \times 5 = 1 \times 5 = 5 \text{V} \]
2.4. Resistor Combination

2.4.1. Series resistors:

(a) Meaning: From the point of view of “circuit 1”, the two resistors $R_1$ and $R_2$ which are connected in series can be replaced by an equivalent resistor $R_{eq}$ where $R_{eq} = R_1 + R_2$.

(b) The use of the symbol “$\equiv$” means that the two circuits are “equivalent” with respect to (wrt.) the two terminals a-b.

2.4.2. Parallel resistors:

Example 2.4.3. $6\parallel 3 = \frac{6 \times 3}{6+3} = \frac{18}{9} = 2$

Example 2.4.4. $R\parallel R = \frac{\frac{R}{R+R}}{R\parallel R} = \frac{\frac{R}{2R}}{R\parallel R} = \frac{1}{2}$

Example 2.4.5. $(R\parallel nR) = \frac{R\parallel nR}{R+nR} = \frac{\frac{R}{R+R}}{R\parallel nR} = \frac{n}{1+n} R$

Example 2.4.6. $(mR)\parallel (nR) = \frac{mR\parallel nR}{mR+nR} = \frac{\frac{mR}{mR+nR}}{mR+nR} = \frac{m+n}{m+n} = R (m\parallel n)$
2.4.7. Generalization to $n$ resistors:

$$R_{eq} = \frac{1}{\sum_{k=1}^{n} \frac{1}{R_k}}$$

These circuits are equivalent with circuit 1.

Example 2.4.8. Find $R_{eq}$ for the following circuit.

Example 2.4.9. Caution: Equivalence is always with respect to some two terminals.

- To reduce confusion, it is important to specify these terminals.
- However, in many cases, the terminals are not explicitly specified because it is quite “clear” where they are in the circuit.

Example 2.4.10. Find the equivalent resistance of the following circuit.
2.4.11. Later in this class, we will have to deal with many resistor-combination scenarios which have “hanging” branches or “dangling” branches. This is demonstrated in the next example.

When working on such problem, some resistors can be eliminated from the circuits. When it is not clear whether a particular resistor can be removed, we can “try connecting a 1A current source” across the two terminals and see whether there is any current flowing through that resistor.

Example 2.4.12. Find $R_{eq}$ of the following circuit

2.4.13. An advanced-level trick for finding $R_{eq}$: Connect a 1A current source across the two terminals and (use other techniques to) solve for the value $V$ of the voltage across the two terminals (under the 1A current source), then

To find $R_{eq}$:

Ohm’s law: $V = i \times R_{eq}$

the Ohm’s law implies that we must have

$$R_{eq} = \frac{V}{1} = V.$$

In other words, the voltage will be exactly the same as the equivalent resistance. Later on, we will study a seemingly impossible-to-solve resistor-combination question using such observation.
Example 2.4.14. A revisit to Example 2.4.12.

2.4.15. Consider the circuit below.
(a) Find the equivalent resistance with respect to terminals a-c.
(b) Find the equivalent resistance with respect to terminals c-d.
2.5. Dividers

2.5.1. **Voltage divider**: When two resistors are connected in *series*, the “total” voltage across them is *divided* among the resistors in *direct proportion* to their resistances.

![Voltage Divider Diagram]

Caution: Before applying this formula, make sure that the same current passes through both resistors. (No extra current leaks in or out at node c.)

- This means the formula still works when there is/are “hanging” branch(es) at node c.

**Example 2.5.2.** Find $V_0$ in the following circuit.

![Example 2.5.2 Circuit Diagram]

$V_0 = \frac{2}{2+3} \times 10 = 4 \text{ V}$

**Example 2.5.3.** Find $V_0$ in the following circuit.

![Example 2.5.3 Circuit Diagram]

$V_0 = \frac{2}{2+3} \times (-10) = -4 \text{ V}$
2.5.4. Generalization:

\[
I_T = \frac{1}{R_1} + \frac{1}{R_2} \quad I_I = \frac{1}{R_1} + \frac{1}{R_2}
\]

2.5.5. Current divider: When two resistors are connected in parallel, the “total” current through them is divided among the resistors in inverse proportion to their resistances.

Example 2.5.6.

\[
I_T = \frac{1}{R_1} + \frac{1}{R_2}
\]

Assume \( R_T = 0 \)

\[
I_T = 0 A
\]

2.5.7. Generalization:

\[
I_T = \frac{1}{R_1} + \frac{1}{R_2} \quad I_T = 0 A
\]

\[
I_T = 1 MA.
\]
Example 2.5.8. Find $i_o, v_o, p_o$ (power dissipated from the $3\Omega$ resistor).

Example 2.5.9. Three light bulbs are connected to a 9V battery as shown below. Calculate: (a) the total current supplied by the battery, (b) the current through each bulb, (c) the resistance of each bulb.
2.7. Measuring Devices

2.6. Practical Voltage and Current Sources

An ideal voltage source is assumed to supply a constant voltage. This implies that it can supply very large current even when the load resistance is very small.

However, a practical voltage source can supply only a finite amount of current. To reflect this limitation, we model a practical voltage source as an ideal voltage source connected in series with an internal resistance $r_s$, as follows:

Similarly, a practical current source can be modeled as an ideal current source connected in parallel with an internal resistance $r_s$.

2.7. Measuring Devices

Ohmmeter: measures the resistance of the element. Important rule: Measure the resistance only when the element is disconnected from circuits.

Ammeter: connected in series with an element to measure current flowing through that element. Since an ideal ammeter should not restrict the flow of current, (i.e., cause a voltage drop), an ideal ammeter has zero internal resistance.

Voltmeter: connected in parallel with an element to measure voltage across that element. Since an ideal voltmeter should not draw current away from the element, an ideal voltmeter has infinite internal resistance.
Instructions
i. Separate into groups of no more than three persons.
ii. Only one submission is needed for each group. Late submission will not be accepted.
iii. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
iv. Do not panic.

1. Find $I_1$ when $I_S = 10\, \text{A}$, $R_1 = 3\, \text{k}\Omega$ and $R_2 = 2\, \text{k}\Omega$.

\[ I_S \quad R_1 \quad I_1 \quad R_2 \quad I_2 \]

2. Consider the circuit below.

\begin{center}
\begin{tabular}{|c|c|}
\hline
Name & ID \\
\hline
\end{tabular}
\end{center}

a. Find the equivalent resistance with respect to terminals a-b

b. Find the equivalent resistance with respect to terminals b-c

c. Find the equivalent resistance with respect to terminals a-c
CHAPTER 3

Methods of Analysis

Here we apply the fundamental laws of circuit theory (Ohm’s Law & Kirchhoff’s Laws) to develop two powerful techniques for circuit analysis.

1. Nodal Analysis (based on KCL)
2. Mesh Analysis (based on KVL)

This is the most important chapter for our course.

3.1. Nodal Analysis

Here, we analyze circuit using node voltages as the circuit variables.

**Example 3.1.1.** Back to Example 2.5.8

1. There are three nodes in this circuit
2. Use/select node b as the reference node
3. \( V_c = 12 \, \text{V} \)
4. KCL @ \( \alpha \):
\[
\frac{V_a - 12}{4} + \frac{V_a - 0}{6} + \frac{V_a - 0}{3} = 0
\]
5. \( \Rightarrow V_a = 4 \, \text{V} \)
Review: So far, two kinds of voltages

1. (Regular) Voltages defined using two nodes
   Ex. $V_{ab}$ = the potential difference between node $a$ and node $b$

   \[ a \quad \text{----} \quad b \]

   \[ + \quad V_{ab} \quad - \]

2. Node voltages defined using only one node
   Ex. $V_a$ = the voltage at node "a"
   - The second node is implicit

   - Assumed to be the ground node \( g \) \( \frac{1}{\text{ref. node}} \)

   - Therefore, when we write $V_a$
     we mean $V_{ag}$

Now that we define node voltages, we can write

\[
V_{ab} = V_a - V_b \\
V_{ba} = V_b - V_a
\]

KVL:

\[
V_{ag} - V_{ab} - V_{bg} = 0 \\
V_{ab} = V_{ag} - V_{bg} = V_a - V_b
\]

One more thing ... about KVL

KVL

\[
A \rightarrow B \rightarrow C \rightarrow D \rightarrow A \\
( - V_{ab} - V_{bc} - V_{cd} - V_{da} - (V_b - V_c - V_d - V_a)) = 0
\]
3.1.2. **Steps to Determine Node Voltages:**

**Step 0:** Determine the number of nodes $n$.

**Step 1:** Select a node as a reference node (ground node). Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining $n - 1$ nodes.
- The voltage are now referenced with respect to the reference node.
- The ground node is assumed to have 0 potential.

- Recall that voltages are measured between two points. For node voltages, the second point is always the ground node.

**Step 1s:** If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node “equal” to the voltage of the voltage source.

**Step 2:** Apply KCL to each of the remaining nonreference nodes:
(a) Use Ohm’s law to express the branch currents in terms of the node voltages.

(b) Current source automatically gives current value.

Caution: Watch out for the direction of the arrow.

**Step 3:** Solve the resulting simultaneous equations to obtain the unknown node voltages.

**Step 4:** After all the node voltages are determined, it is easy to find everything else.
3.1.3. Remarks: There are multiple methods to solve the simultaneous equations in Step 3.

- Method 1: Elimination technique (good for a few variables)
- Method 2: Write in term of matrix and vectors (write $Ax = b$), then use Cramer’s rule.
- Method 3: Use
  - computer (e.g., MATLAB) to find $A^{-1}$ and then find $x = A^{-1}b$
  - calculator (fx-991MS can solve simultaneous linear equations with two or three unknowns.)

**Example 3.1.4.** Calculate the node voltages in the circuit below.

**Example 3.1.5.** Calculate the node voltages in the circuit below.
The next example shows that the steps given in 3.1.2 are not sufficient to solve all interesting circuit-analysis problem.

**Example 3.1.6.** Find $v$ and $i$ in the circuit below.

![Circuit diagram](image)

3.1.7. An extra step should be added to 3.1.2:

**Step 1sn:** If there is a voltage source connected between two nonreference nodes, the two nonreference nodes form a **supernode**. We apply both KCL and KVL to determine the node voltages.

3.1.8. Note the following properties of a supernode:

(a) The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.

(b) A supernode has no voltage of its own.

(c) We can have more than two nodes forming a single supernode.

(d) The supernodes are treated differently because nodal analysis requires knowing the current through each element. However, there is no way of knowing the current through a voltage source in advance.
3.2. Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables.

**Definition 3.2.1.** Mesh is a loop which does not contain any other loop within it.

3.2.2. Steps to Determine Mesh Currents:

**Step 0:** Determine the number of meshes $n$.

**Step 1:** Assign mesh currents $i_1, i_2, \ldots, i_n$, to the $n$ meshes.
   - The direction of the mesh current is arbitrary—(clockwise or counterclockwise)—and does not affect the validity of the solution.
   - For convenience, we define currents flow in the clockwise (CW) direction.

**Step 2a:** From the current direction in each mesh, define the voltage drop polarities.

**Step 2b:** Apply KVL to each of the $n$ meshes.

Use Ohm’s law to express the voltages in terms of the mesh current.
   - Tip (for combining Step 2a and 2b): Go around the loop in the same direction as the mesh current (of that mesh). When we pass a resistor $R$, the voltage drops by $I \times R$ where $I$ is the branch current (algebraic sum of mesh currents) through that resistor in the mesh-current direction.

**Step 3:** Solve the resulting $n$ simultaneous equations for the mesh currents.

**Step 4:** Other quantities related to the circuit can be found from the mesh currents.

---

1 Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously.
2 Note that according to our agreement above, this is in the CW direction.
3. METHODS OF ANALYSIS

Example 3.2.3. Back to Example 2.5.8

KVL around Mesh 1:

\[ 12 - i_1 4 - (i_1 - i_2) 6 = 0 \]

\[ +12 - V_a - V_b = 0 \]

KVL around Mesh 2:

\[ -(i_2 - i_1) 6 - i_2 3 = 0 \]

Calc. \( i_1 = 2 \) A

\( i_2 = \frac{4}{3} \) A

\( V_0 = i_2 3 = \frac{4}{3} \times 3 = 4 \) V

Example 3.2.4. Find the branch currents \( I_1, I_2, \) and \( I_3 \) using mesh analysis.

KVL around mesh 1:

\[ 15 - i_1 5 - 10(i_1 - i_2) - 10 = 0 \]

\[ 3i_1 - 2i_2 = 1 \]

\[ 15 \text{ V} \]

Calc. \( i_1 = 1 \) A

\( i_2 = 1 \) A

KVL around mesh 2:

\[ 10 - (i_2 - i_1) 10 - 6i_2 - 4i_2 = 0 \]

\[ i_1 - 2i_2 = -1 \]

\[ 10 \text{ V} \]

\[ I_1 = i_1 = 1 \text{ A} \]

\[ I_2 = i_2 = 1 \text{ A} \]

\[ I_3 = i_1 - i_2 = 1 - 1 = 0 \text{ A} \]
**Nodal Analysis**

- **Create**
  - one reference node (\(\bullet\))
  - other nodes are non-ref. nodes.

  ![Nodal Analysis Diagram](image)

- **Define node voltage at each node.**
  - The node voltage at the ref. node is 0.
  - All other node voltages are w.r.t. to the reference node.

- **Unknown variables**
  - node voltages: \(V_a, V_b, \ldots\)
  - mesh currents: \(i_1, i_2, \ldots\)

- **Underlying principles**
  - **KCL + Ohm’s Law**
    - \(\sum\) of current (outgoing \(\rightarrow\) + incoming \(\rightarrow\) -) at each node = 0
  - **KVL + Ohm’s Law**
    - \(\sum\) of voltage (rise \(\rightarrow\) + drop \(\rightarrow\) -) around each mesh (loop) = 0

- **Easy cases**
  - Easy node voltage:
    - A voltage source connecting the ground and a non-ref. node.

  - The node voltage here is \(V_a\).

  - The node voltage here is \(-V_a\).

- **Apply the principles**
  - at each remaining node

  - around each remaining mesh

- **Ohm’s law**
  - The amount of current going from node A to node B is \(V_a - V_b\).
  - The amount of current going from node B to node A is \(V_b - V_a\).

- **More Easy cases**
  - Outgoing current = \(+I_s\)
  - Outgoing current = \(-I_s\)

**Mesh Analysis**

- **Mesh currents**
  - In our class, all of them are in the clockwise direction.

- **Remark:** Need to be combined to get the actual current in a branch.

- **Easy mesh current:**
  - A source with only one mesh current passing through it.

  - \(i_{k} = I_s\)
  - \(i_{k} = -I_s\)

- **Ohm’s law**
  - Moving around the kth mesh in the clockwise direction, when we pass \(R\), the voltage drops by the amount \((i_{k} - i_{m})R\).
3.3. Remarks on Nodal Analysis and Mesh Analysis

3.3.1. Nodal analysis applies KCL to find unknown (node) voltages in a given circuit, while mesh analysis applies KVL to find unknown (mesh) currents.

3.3.2. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is \textit{planar}.
- A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is nonplanar.

3.3.3. \textbf{Nodal Analysis vs. Mesh Analysis}: Given a network to be analyzed, how do we know which method is better or more efficient?

Suggestion: You should be familiar with both methods. Choose the method that results in smaller number of variables or equations.
- A circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis.

You can also use one method to check your results of the other method.

3.3.4. Nodal analysis and mesh analysis can also be used to find equivalent resistance of a part of a circuit.

This becomes extremely useful when the techniques that we studied in the previous chapter cannot be directly applied (e.g., when we can’t find resistors that are in parallel or in series; they are all connected in some “strange” configuration.)

The are two approaches to this kind of problems.

(a) Apply 1 V voltage source across the terminals, find the corresponding current $I$ through the voltage source. Then,

$$R_{eq} = \frac{V}{I} = \frac{1}{I}.$$ 

(b) Put 1 A current source through the terminals, find the corresponding voltage $V$ across the current source. Then,

$$R_{eq} = \frac{V}{I} = \frac{V}{1} = V.$$
Example 3.3.5. Find the equivalent resistance for the following circuit

Good: Use Node Analysis to find $V_1$

$V_{ab} = V_1$

$\text{KCL @ node } 1$

$-1 + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{1} + \frac{1A}{1} = 0$

$\text{KCL @ node } 2$

$\frac{V_2}{1} + \frac{V_3 - V_5}{1} + \frac{V_5}{1} = 0$

$\text{KCL @ node } 3$

$\frac{V_3 - V_4}{1} + \frac{V_4 - V_5}{1} + \frac{V_5}{1} = 0$

calc.

$\Rightarrow V_1 = 1V \Rightarrow V_{ab} = 1V \Rightarrow R_{eq} = 1 \Omega$
CHAPTER 4

Circuit Theorems

The growth in areas of application of electrical circuits has led to an evolution from simple to complex circuits. To handle such complexity, engineers over the years have developed theorems to simplify circuit analysis. These theorems (Thevenin’s and Norton’s theorems) are applicable to linear circuits which are composed of resistors, voltage and current sources.

**Definition 4.0.6. System:**

\[ \text{Input} \rightarrow S \rightarrow \text{Output} \]

**4.1. Linearity Property**

**Definition 4.1.1.** A linear system is a system whose output is linearly related (or directly proportional) to its input. In particular, when we say that the input and output are linearly related, we mean they need to satisfy two properties:

(a) Homogeneous (Scaling): If the input is multiplied by a constant \( k \), then we should observe that the output is also multiplied by \( k \).

\[ S(kx) = kS(x) \]

(b) Additive: If the inputs are summed then the outputs are summed.

\[ S(x_1 + x_2) = S(x_1) + S(x_2) \]

**Example 4.1.2.** Is the function \( f(x) = x^2 + 1 \) linear?

**Check:**

(a) \( f(kx) = kx^2 + 1 \), for all \( k \),

\[ (kx)^2 + 1 = k(x^2 + 1) \]

(b) \( f(x_1 + x_2) = (x_1 + x_2)^2 + 1 \),

\[ (x_1 + x_2)^2 + 1 \neq x_1^2 + 1 + x_2^2 + 1 \]

**Conclusion:** \( f \) is not linear.

---

\( ^1 \) The input and output are sometimes referred to as cause and effect, respectively.
4. CIRCUIT THEOREMS

Example 4.1.3. Is the function \( f(x) = 3x + 1 \) linear?

Conclusion:
- \( f(x) = 3x \) is \textit{not} linear
- \( f(x+1) \neq f(x) + f(1) \)

4.1.4. A \textbf{one-dimensional linear function} is a function of the form

\[ y = ax \]

for some constant \( a \).

- For a system, we may call it a \textit{single-input single-output} (SISO) system.
- In radio it is the use of only one antenna both in the transmitter and receiver.

4.1.5. A \textbf{multi-dimensional linear function} is a function of the form

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = \mathbf{A}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

for some matrix \( \mathbf{A} \).

- For a system, when both \( m \) and \( n \) are greater than one, we may call it a \textit{multiple-input multiple-output} system (MIMO) system.

\text{MISO: } \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{(linear combination)}

\text{SIMO: } \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} \gamma_i \\ \vdots \\ \gamma_n \end{pmatrix}

- When \( m = n = 1 \), we are back to the one-dimensional case in 4.1.4.

Example 4.1.6. A resistor is a \textbf{linear element} when we consider the current \( i \) as its input and the voltage \( v \) as its output.

4.1.7. For us, we are considering \textbf{linear circuit}.

- The \( x_j \)'s will be \textbf{all} of the (independent) voltage and current \textbf{sources} in the circuit.

- Each \( y_j \) will correspond to \textbf{the current or voltage value under investigation}.
Example 4.1.8. The circuit below is excited by a voltage source $v_s$, which serves as the input. Assume that the circuit is linear.

The circuit is terminated by a load $R$. We take the current $i$ through $R$ as the output. Suppose $v_s = 10$ V gives $i = 2$ A. By the assumed linearity, $v_s = 1$ V will give $i = 0.2$ A. By the same token, $i = 1$ mA must be due to $v_s = 5$ mV.

Example 4.1.9. For the circuit below, find $v_o$ when (a) $i_s = 15$ and (b) $i_s = 30$.

4.1.10. Because $p = i^2 R = v^2 / R$ (making it a quadratic function rather than a linear one), the relationship between power and voltage (or current) is nonlinear. Therefore, the theorems covered in this chapter are not applicable to power.
4.2. Superposition

Example 4.2.1. Find the voltage $v$ in the following circuit.

From the expression of $v$, observe that there are two contributions.

(a) When $I_s$ acts alone (set $V_s = 0$),

$$v = \frac{R_1 R_2}{R_1 + R_2} I_s$$

(b) When $V_s$ acts alone (set $I_s = 0$),

$$v = \frac{R_2}{R_1 + R_2} V_s$$

Key Idea: Find these contributions from the individual sources and then add them up to get the final answer.

Definition 4.2.2. Superposition technique is a way to determine currents and voltages in a circuit that has multiple independent sources by considering the contribution of one source at a time and then add them up.

4.2.3. The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

- So, if the circuit has $n$ sources, then we have $n$ cases: “source 1 acting alone”, “source 2 acting alone”, . . . , “source n acting alone”.
4.2.4. To apply the superposition principle, we must keep two things in mind.

(a) We consider one independent source at a time while all other independent sources are \textit{turned off}\footnote{Other terms such as killed, made inactive, deadened, or set equal to zero are often used to convey the same idea.}.

- Replace other independent voltage sources by 0 V (or \textit{short circuits})
- Replace other independent current sources by 0 A (or \textit{open circuits})

This way we obtain a simpler and more manageable circuit.

(b) Dependent sources are left intact because they are controlled by circuit variable.

4.2.5. \textbf{Steps to Apply Superposition Principles:}

\textbf{S1:} Turn off all independent sources except one source.
Find the output due to that active source. (Here, you may use any technique of your choice.)

\textbf{S2:} Repeat S1 for each of the other independent sources.

\textbf{S3:} Find the total contribution by adding algebraically all the contributions due to the independent sources.

\textbf{Example 4.2.6.} Back to Example 4.2.1
Example 4.2.7. Using superposition theorem, find $v_o$ in the following circuit.

4.2.8. Remark on linearity: Keep in mind that superposition is based on linearity. Hence, we cannot find the total power from the power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current and hence it is not linear (e.g. because $5^2 \neq 1^2 + 4^2$).

4.2.9. Remark on complexity: Superposition helps reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

However, it may very likely involve more work. For example, if the circuit has three independent sources, we may have to analyze three circuits. The advantage is that each of the three circuits is considerably easier to analyze than the original one.
4.3. Source Transformation

We have noticed that series-parallel resistance combination helps simplify circuits. The simplification is done by replacing one part of a circuit by its equivalence. Source transformation is another tool for simplifying circuits.

4.3.1. A source transformation is the process of replacing a voltage source in series with a resistor $R$ by a current source in parallel with a resistor $R$ or vice versa.

Notice that when terminals $a - b$ are short-circuited, the short-circuit current flowing from $a$ to $b$ is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the righthand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_sR \quad \text{or} \quad i_s = \frac{v_s}{R}.$$

\footnote{Recall that an equivalent circuit is one whose $v-i$ characteristics are identical with the original circuit.}
4.3.2. Source transformation is usually used repeatedly in combination with the “source combination” and “resistor combination” techniques studied in earlier chapter. In our class, when we say “use source transformation”, we actually mean to use all the three techniques above repeatedly to simplify the circuit. At the end, the unknown current or voltage value can usually be obtained by the current divider formula or the voltage divider formula, respectively.

Example 4.3.3. Use source transformation to find $v_0$ in the following circuit:
4.3.4. Cautions:

(a) The “$R$” in series with the voltage source and the “$R$” in parallel with the current source are not the same “resistor” even though they have the same value. In particular, the voltage values across them are generally different and the current values through them are generally different.

(b) Keep a circuit variable as a fixed point in the circuit. Do not blindly “transform” and “combine”.

(c) From (4.2), an ideal voltage source with $R = 0$ cannot be replaced by a finite current source. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.
4.4. Thevenin’s Theorem

4.4.1. It often occurs in practice that a particular element in a circuit or a particular part of a circuit is variable (usually called the load) while other elements are fixed.

- As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load.

Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin’s theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

4.4.2. Thevenin’s Theorem is an important method to simplify a complicated circuit to a very simple circuit. It states that a circuit can be replaced by an equivalent circuit consisting of an independent voltage source $V_{Th}$ in series with a resistor $R_{Th}$, where

- $V_{Th}$: the open-circuit voltage at the terminals,
- $R_{Th}$: the equivalent resistance at the terminals when the independent sources are turned off.

This theorem allows us to convert a complicated network into a simple circuit consisting of a single voltage source and a single resistor connected in series. The circuit is equivalent in the sense that it looks the same from the outside, that is, it behaves the same electrically as seen by an outside observer connected to terminals a and b.
4.4.3. **Steps to Apply Thevenin’s theorem.**

**S1:** Find $R_{Th}$: Turn off all independent sources. $R_{Th}$ is the equivalent resistance of the network looking between terminals $a$ and $b$.

**S2:** Find $V_{Th}$: Open the two terminals (remove the load) which you want to find the Thevenin equivalent circuit. $V_{Th}$ is the open-circuit voltage across the terminals.

**S3:** Connect $V_{Th}$ and $R_{Th}$ in series to produce the Thevenin equivalent circuit for the original circuit.

**Example 4.4.4.** Find the Thevenin equivalent circuits of each of the circuits shown below, to the left of the terminals a-b.
Example 4.4.5. Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals a-b. Then find the current through $R_L = 6$, 16, and 36 Ω.

Solution:

$$R_{Th} = \left( \frac{4}{1}\right) + 1 = 4 + 1 = 5 \text{Ω}$$

KCL at node $a$:

$$V_x - 32 + \frac{V_x - 0}{12} - 2 = 0$$

$V_x = 30 \text{V}$

$$I_L = \frac{30}{4 + R_L}$$

$R_L$ | $I_L$
---|---
6 | 3 A
16 | 1.75 A
36 | 0.75 A
Example 4.4.6. Determine the current $I$ in the branch a-b in the circuit below.

There are many approaches that we can take to obtain the current $I$. For example, we could apply nodal analysis and determine the node voltages at nodes $a$ and $b$ and thereby determine the current $I$ by Ohm’s law. However, here, we will find the Thévenin equivalent circuit for the subcircuit to the left of the $aa'$ terminal pair (Subcircuit A) and for the subcircuit to the right of the $bb'$ terminal pair (Subcircuit B), and then using these equivalent subcircuits to find the current $I$. 

\[ I = \frac{2}{2+1+2} = \frac{2}{5} \text{ A} \]
4. CIRCUIT THEOREMS

4.5. Norton’s Theorem

Norton’s Theorem gives an alternative equivalent circuit to Thevenin’s Theorem.

4.5.1. Norton’s Theorem: A circuit can be replaced by an equivalent circuit consisting of a current source $I_N$ in parallel with a resistor $R_N$, where $I_N$ is the short-circuit current through the terminals and $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

Note: $R_N = R_{TH}$ and $I_N = \frac{V_{TH}}{R_{TH}}$. These relations are easily seen via source transformation.\(^4\)

4.5.2. Steps to Apply Norton’s Theorem

S1: Find $R_N$ (in the same way we find $R_{TH}$).

S2: Find $I_N$: Short circuit terminals $a$ to $b$. $I_N$ is the current passing through $a$ and $b$.

S3: Connect $I_N$ and $R_N$ in parallel.

\(^4\)For this reason, source transformation is often called Thevenin-Norton transformation.
Example 4.5.3. Back to the circuit in Example 4.4.5. Directly find the Norton equivalent circuit of the circuit shown below, to the left of the terminals a-b.

Remark: In our class, to “directly find the Norton equivalent circuit” means to follow the steps given in 4.5.2. Similarly, to “directly find the Thevenin equivalent circuit” means to follow the steps given in 4.4.3.

Suppose there is no requirement to use the direct technique, then it is quite easy to find the Norton equivalent circuit from the derived Thevenin equivalent circuit in Example 4.4.5.
Example 4.5.4. Directly find the Norton equivalent circuit of the following circuit at terminals a-b.

![Circuit Diagram](image1.png)

$$I_N = \frac{V_S}{R} + I_S$$

Example 4.5.5. Directly find the Norton equivalent circuit of the circuit in the following figure at terminals a-b.

![Circuit Diagram](image2.png)

**Norton Equivalent Circuit:**

$$R_N = \text{eq. resistance when all sources are deactivated}$$

$$R_N = (8+4+8)/5 = 20/5 = 4 \Omega$$

$$V_R = 12 + \frac{V_n - 12}{4} + \frac{V_n - 0}{16} = 0$$

$$\Rightarrow V_n = 16 \text{V}$$

$$I_n = \frac{V_n - 0}{16} = 1 \text{A}$$

$$I_{sc} = \frac{V_n - 0}{16} = 1 \text{A}$$

With the short-circuit addition, node a and node b are now the same node.

This 5 Ω can be ignored because it is in parallel with a short circuit. (OV across ⇒ 0A through)
4.6. Maximum Power Transfer

In many practical situations, a circuit is designed to provide power to a load. In areas such as communications, it is desirable to maximize the power delivered to a load. We now address the problem of delivering the maximum power to a load when given a system with known internal losses.

4.6.1. Questions:

(a) How much power can be transferred to the load under the most ideal conditions?

(b) What is the value of the load resistance that will absorb the maximum power from the source?

4.6.2. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown below, the power delivered to the load resistor \( R_L \) is

\[
p = i^2 R_L \quad \text{where} \quad i = \frac{V_{th}}{R_{th} + R_L}
\]

The derivative of \( p \) with respect to \( R_L \) is given by

\[
\frac{dp}{dR_L} = 2i \frac{di}{dR_L} R_L + i^2 = 2 \frac{V_{th}}{R_{th} + R_L} \left( - \frac{V_{th}}{(R_{th} + R_L)^2} \right) R_L + \left( \frac{V_{th}}{R_{th} + R_L} \right)^2
\]

\[
= \left( \frac{V_{th}}{R_{th} + R_L} \right)^2 \left( - \frac{2R_L}{R_{th} + R_L} + 1 \right)
\]

We then set this derivative equal to zero and get

From calculus, \( R_L^* = R_{TH} \).

\[
P_{\text{max}} = i^2 R_L = \left( \frac{V_{TH}}{2R_{TH}} \right)^2 R_{TH} = \frac{V_{TH}^2}{4 R_{TH}}
\]
4.6.3. **Maximum power transfer occurs** when the load resistance \( R_L \) equals the Thevenin resistance \( R_{TH} \). The corresponding maximum power transferred to the load \( R_L \) equals to

\[
p_{\text{max}} = \left( \frac{V_{TH}}{R_{TH} + R_{TH}} \right)^2 R_{TH} = \frac{V_{TH}^2}{4 R_{TH}}.
\]

**Example 4.6.4.** Connect a load resistor \( R_L \) across the circuit in Example 4.4.4. Assume that \( R_1 = R_2 = 14\Omega \), \( V_s = 56V \), and \( I_s = 2A \). Find the value of \( R_L \) for maximum power transfer and the corresponding maximum power.

\[
V_{TH} = V_s - R_1 I_s = 56 - 29 = 27V
\]

\[
p_{\text{max}} = \frac{V_{TH}^2}{4 R_{TH}} = \frac{(27 - 14 \times 2)^2}{4 \times 14} = \frac{28^2}{2 \times 28} = 14W
\]

**Example 4.6.5.** Find the value of \( R_L \) for maximum power transfer in the circuit below. Find the corresponding maximum power.