Sirindhorn International Institute of Technology
Thammasat University
School of Information, Computer and Communication Technology

Lecture Notes:
ECS 203 Basic Electrical Engineering
Semester 1/2010

Dr.Prapun Suksompong[1]

June 16, 2010

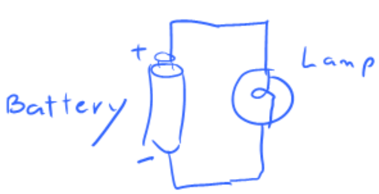
[1]Special thanks to Dr.Waree Kongprawechnon and Dr.Somsak Kittipiyakul for earlier versions of the notes. Parts of the notes were compiled from C.K. Alexander and M.N.O. Sadiku, *Fundamentals of Electric Circuits*, 4th ed., McGraw-Hill, International Edition, 2009 and G. Rizzoni, *Principles and Applications of Electrical Engineering*, 5th ed., Mc-Graw-Hill, International Edition, 2007

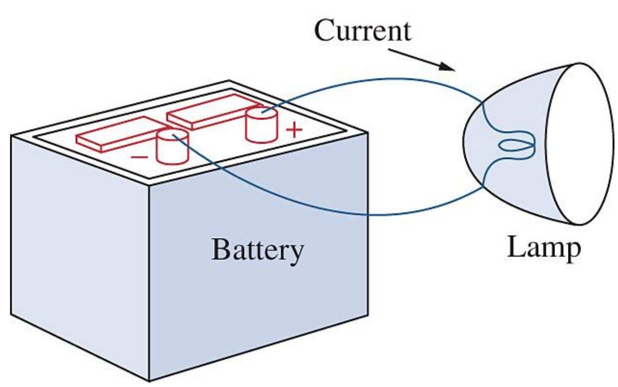
CHAPTER 1

Basic Concepts

In electrical engineering, we are often interested in communicating or transferring energy from one point to another. To do this requires an interconnection of electrical devices. Such interconnection is referred to as an **electric circuit**, and each component of the circuit is known as an **element**.

DEFINITION 1.0.1. An **electric circuit** is an interconnection of electrical elements.





1.1. Systems of Units

1.1.1. As engineers, we deal with measurable quantities. Our measurement must be communicated in standard language that virtually all professionals can understand irrespective of the country. Such an international measurement language is the **International System of Units** (SI).

- In this system, there are six principal units from which the units of all other physical quantities can be derived.

Quantity	Basic Unit	Symbol
Length	meter	m
Mass	kilogram	Kg
Time	second	s
Electric Current	ampere	A
Temperature	kelvin	K
Luminous Intensity	candela	cd
Charge	coulomb	C

- One great advantage of SI unit is that it uses prefixes based on the power of 10 to relate larger and smaller units to the basic unit.

Current

5 A

500 A

5000 A = 5 kA

Multiplier	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
$\cancel{10^{-2}}$	~~centi~~	~~c~~
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

EXAMPLE 1.1.2. Change of units:

$$600,000,000 \text{ m}\cancel{\text{m}}^{A} = 6 \times 10^{8} \times 10^{-3} \text{ A} = 6 \times 10^{5} \text{ A}$$

1 2 3 4 5 6 7 8 8 zeros m

$$= 6 \times \frac{10^{6}}{10} \text{ A} = \frac{6}{10} \text{ MA} = 0.6 \text{ MA}$$

1.2. Circuit Variables

8

1.2.1. **Charge**: The concept of electric charge is the underlying principle for all electrical phenomena. Charge is an electrical property of the atomic particles of which matter consists, measured in coulombs (C). The charge of an electron is -1.602×10^{-19} C.

- The coulomb is a large unit for charges. In 1 C of charge, there are $1/(1.602 \times 10^{-19}) = 6.24 \times 10^{18}$ electrons. Thus realistic or laboratory values of charges are on the order of pC, nC, or μC.
- A large power supply capacitor can store up to 0.5 C of charge.

1.2.2. *Law of Conservation of Charge*: Charge can neither be created nor destroyed, only transferred.

DEFINITION 1.2.3. **Current**: The time rate of change of charge, measured in amperes (A). Mathematically,

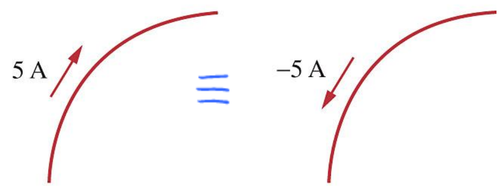
$$i(t) = \frac{d}{dt}q(t)$$

Note:

- 1 ampere (A) = 1 coulomb/second (C/s).
- The charge transferred between time t_1 and t_2 is obtained by

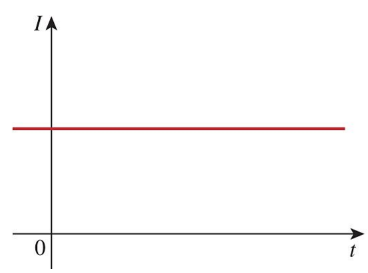
$$q = \int_{t_1}^{t_2} i dt$$

To talk about current, you need to specify two things:
1) direction
2) amount
These are conveyed by
* { a) arrow
b) value (positive/negative)



1.2.4. Two types of currents:

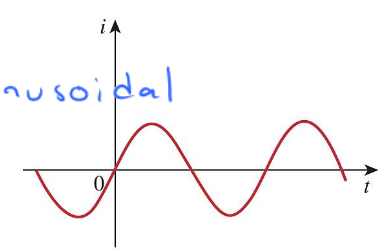
(a) A **direct current** (DC) is a current that remains constant with time.



(b) An **alternating current** (AC) is a current that varies with time.

could mean

sinusoidal



- Such AC current is used in your household, to run the air conditioner, refrigerator, washing machine, and other electric appliances.

1.2.5. By convention the symbol I is used to represent such a constant current. A time-varying current is represented by the symbol i.

Definition 1.2.6. **Voltage** (or potential difference): the energy required to move a unit charge though an element, measured in volts (V). The voltage between two points a and b in a circuit is denoted by v_{ab} and can be interpreted in two ways:

(a) point a is at a potential of v_{ab} volts higher than point b, or
(b) the potential at point a with respect to point b is v_{ab}.

Note:

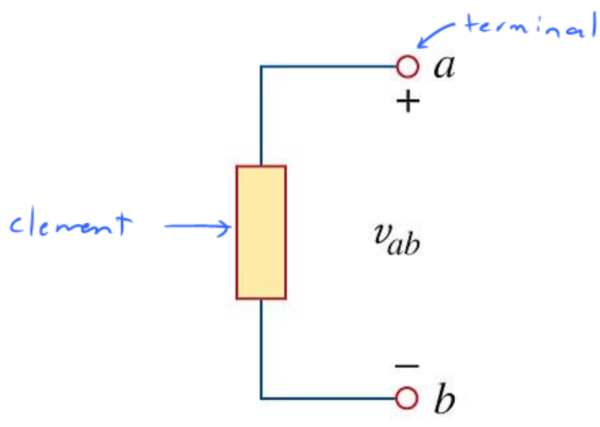
- 1 volt (V) = 1 joule/coulomb = 1 newton-meter/coulomb
- $v_{ab} = -v_{ba}$
- Mathematically,

energy (J)

$$v_{ab} = \frac{dw}{dq}$$

charge (C)

where w is the energy in joules (J) and q is charge in coulombs (C).

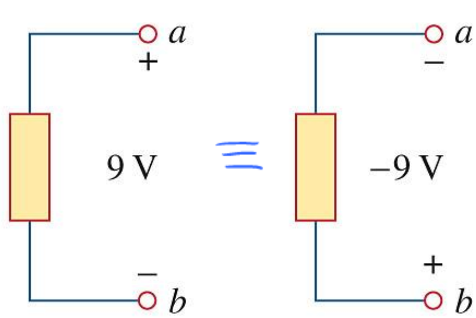


To talk about voltage, you need to specify
a) polarity (+ and −)
b) value (positive or negative number)

The plus (+) and minus (-) signs at the points a and b are used to define reference direction or voltage polarity.

$V_a - V_b = 9\,V$

$V_b - V_a = -9\,V$



1.2.7. Like electric current, a constant voltage is called a **DC voltage** and is represented by V, whereas a sinusoidally time-varying voltage is called an **AC voltage** and is represented by v. A dc voltage is commonly produced by a battery; ac voltage is produced by an electric generator.

Signal = function = wave form

1.2.8. Current and voltage are the two basic variables in electric circuits. The common term ***signal*** is used for an electric quantity such as a current or a voltage (or even electromagnetic wave) when it is used for conveying information. Engineers prefer to call such variables ***signals*** rather than mathematical functions of time because of their importance in communications and other disciplines.

For practical purposes, we need to be able to find/calculate/measure more than the current and voltage. We all know from experience that a 100-watt bulb gives more light than a 60-watt bulb. We also know that when we pay our bills to the electric utility companies, we are paying for the electric energy consumed over a certain period of time. Thus power and energy calculations are important in circuit analysis.

DEFINITION 1.2.9. **Power**: time rate of *expending* or *absorbing* energy, measured in watts (W). Mathematically, the instantaneous power

chain rule

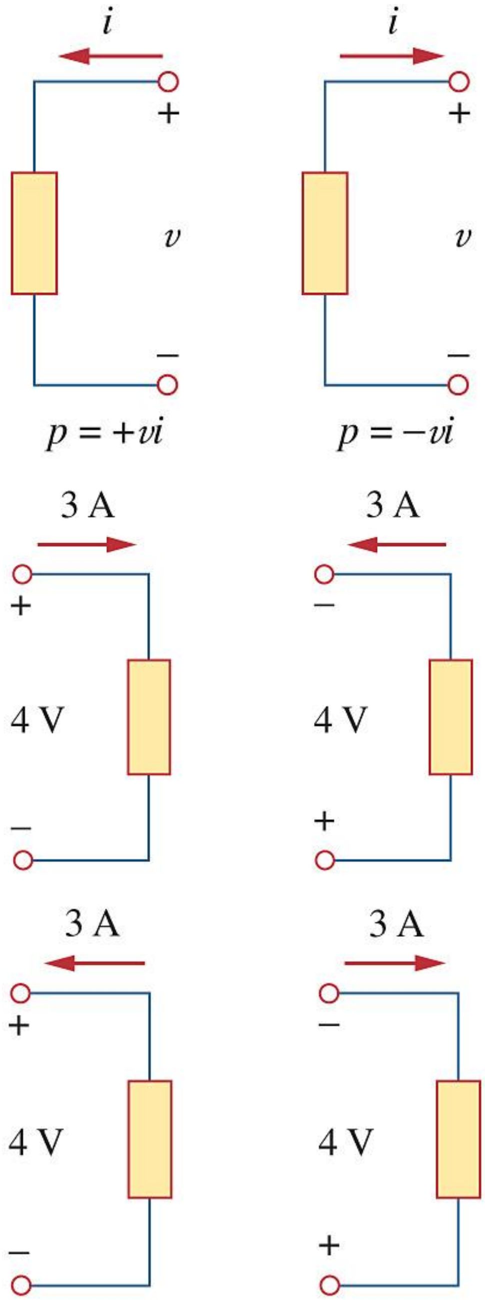
$$p = \frac{dw}{dt} = \frac{dw}{dq}\frac{dq}{dt} = vi$$

DEFINITION 1.2.10. Sign of power

- **Plus sign**: Power is absorbed by the element. (resistor, inductor)
- **Minus sign**: Power is supplied by the element. (battery, generator)

DEFINITION 1.2.11. *Passive sign convention*:

- If the current enters through the positive polarity of the voltage, $p = vi$.
- If the current enters through the negative polarity of the voltage, $p = -vi$.



$p = 3 \times 4 = 12$ W

$3 \times 4 = 12$ W

Absorbing power

$p = -3 \times 4 = -12$ W

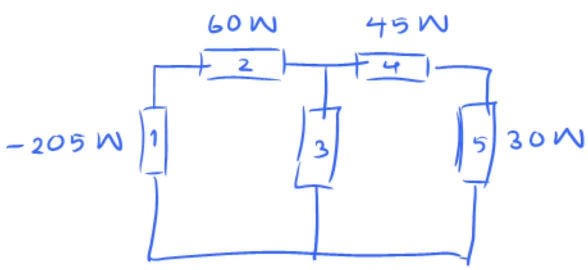
$p = -3 \times 4 = -12$ W

supplying power

Law of Conservation of Energy: Energy can neither be created nor destroyed, only transferred.

- For this reason, the algebraic sum of power in a circuit, at any instant of time, must be zero.
- The total power supplied to the circuit must balance the total power absorbed.

Example 1.2.12.



$P_3 = ?$

$$-205 + 60 + 45 + 30 + P_3 = 0$$

$$P_3 = 205 - 60 - 45 - 30$$

$$= 70\ \mathrm{W}$$

Energy: the energy absorbed or supplied by an element from time 0 to t is

$$w = \int_0^t p\ dt = \int_0^t vi\ dt.$$

- Integration suggests finding area under the curve. Need to be careful with negative area.

The electric power utility companies measure energy in kilowatt-hours (kWh), where 1 kWh = 3600 kJ.

1.3. Circuit Elements

Definition 1.3.1. There are 2 types of elements found in electrical circuits.

1) **Active elements** (is capable of generating energy), e.g., generators, batteries, and operational amplifiers (Op-amp). source

2) **Passive element**, e.g., resistors, capacitors and inductors.

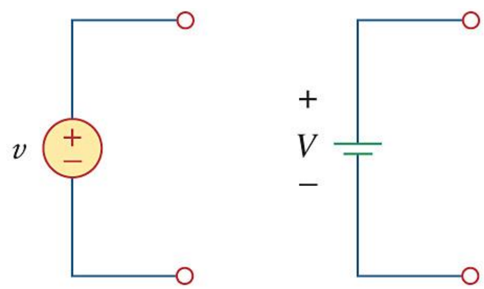
1.3.2. The most important active elements are voltage and current sources:

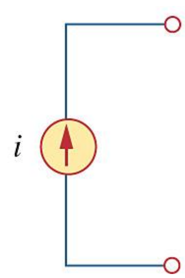
(a) **Voltage source** provides the circuit with a specified voltage (e.g. a 1.5V battery)

(b) **Current source** provides the circuit with a specified current (e.g. a 1A current source).

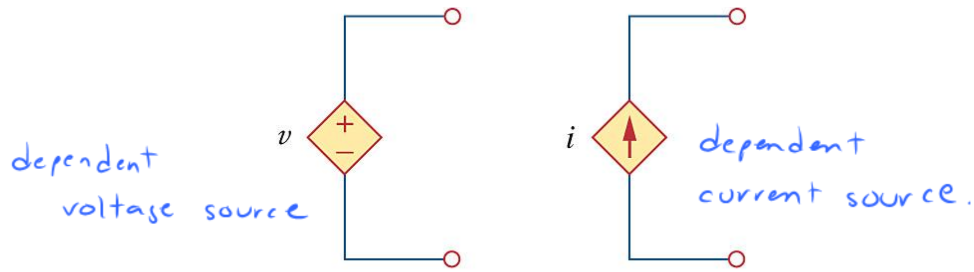
Definition 1.3.3. In addition, we may characterize the voltage or current sources as:

1) **Independent source**: An active element that provides a specified voltage or current that is completely independent of other circuit elements.



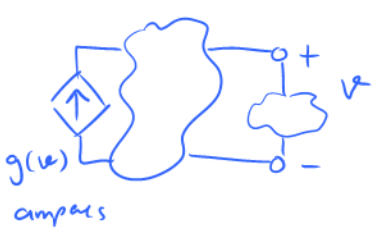


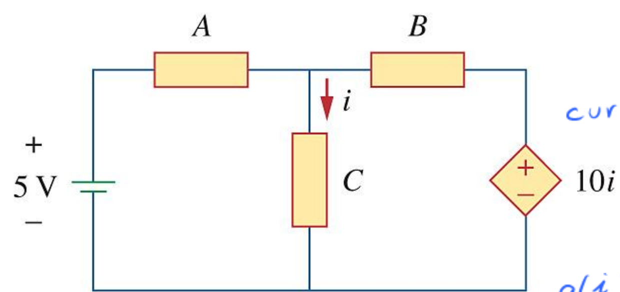
2) **Dependent source**: An active element in which the source quantity is controlled by another voltage or current.



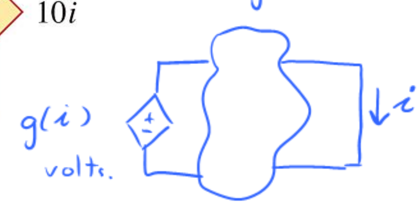
1.3.4. The key idea to keep in mind is that a voltage source comes with polarities (+ -) in its symbol, while a current source comes with an arrow, irrespective of what it depends on.

General form of a voltage-controlled current source





current-controlled voltage source.



1.3.5. Remarks:

- Dependent sources are useful in modeling elements such as transistors, operational amplifiers and integrated circuits.
- An ideal voltage source (dependent or independent) will produce any current required to ensure that the terminal voltage is as stated.
- An ideal current source will produce the necessary voltage to ensure the stated current flow.
- Thus an ideal source could in theory supply an infinite amount of energy.
- Not only do sources supply power to a circuit, they can absorb power from a circuit too.
- For a voltage source, we know the voltage but not the current supplied or drawn by it. By the same token, we know the current supplied by a current source but not the voltage across it.

Review: Last week we've seen

$p = iv$ } Need to check the arrow direction of i and
$v = iR$ } the polarity of v
before applying the formula.

There are only two cases:

① Conform with the passive sign convention:

$p = iv$ $\quad$ $v = iR$

② Not conform with the passive sign convention:

$p = -iv$ $\quad$ $v = -iR$

Extra minus sign

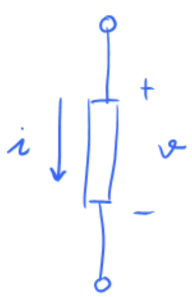
CHAPTER 2

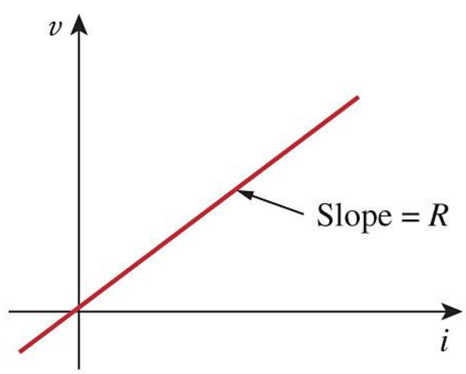
Basic Laws

Here we explore two fundamental laws that govern electric circuits (Ohm's law and Kirchhoff's laws) and discuss some techniques commonly applied in circuit design and analysis.

2.1. Ohm's Law

Ohm's law shows a relationship between voltage and current of a resistive element such as conducting wire or light bulb.



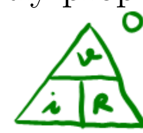


2.1.1. **Ohm's Law**: The voltage v across a resistor is directly proportional to the current i flowing through the resistor.

$$v = iR,$$

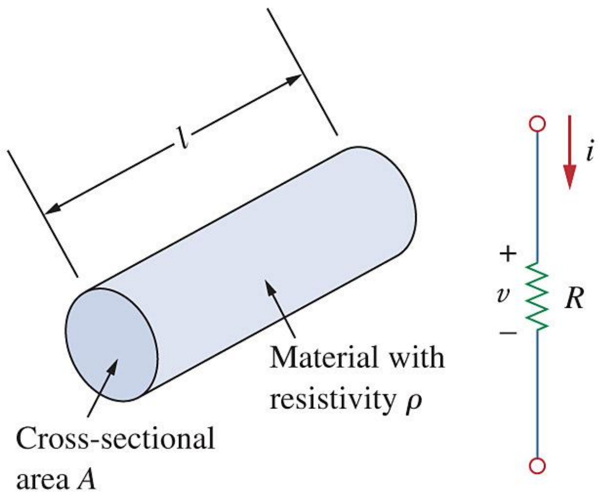
resistance value

Ohm's triangle



where R = resistance of the resistor, denoting its ability to resist the flow of electric current. The resistance is measured in ohms (Ω).

- To apply Ohm's law, the direction of current i and the polarity of voltage v must conform with the passive sign convention. This implies that current flows from a higher potential to a lower potential in order for $v = iR$. If current flows from a lower potential to a higher potential, $v = -iR$.



$v = i \times R$

2.1.2. The resistance R of a cylindrical conductor of cross-sectional area A, length L, and conductivity σ is given by

$$R = \frac{L}{\sigma A}.$$

Alternatively,

$$R = \rho \frac{L}{A}$$

where ρ is known as the resistivity of the material in ohm-meters. Good conductors, such as copper and aluminum, have low resistivities, while insulators, such as mica and paper, have high resistivities.

2.1.3. **Remarks:**

(a) $R = v/i$

(b) Conductance :

$$G = \frac{1}{R} = \frac{i}{v}$$

The unit of G is the mho[1] (℧) or siemens[2] (S)

[1] Yes, this is NOT a typo! It was derived from spelling ohm backwards.

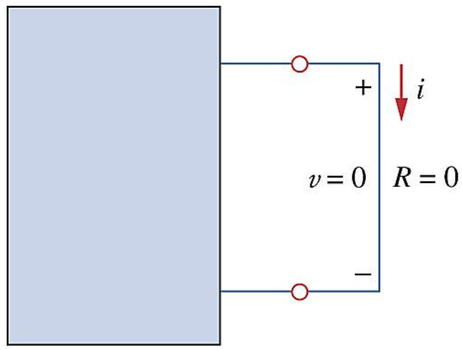
[2] In English, the term siemens is used both for the singular and plural.

(c) The two extreme possible values of R.

(i) When $R = 0$, we have a **short circuit** and

$$v = i\overset{0}{R} = 0$$

showing that $v = 0$ for any i.



voltage difference = 0

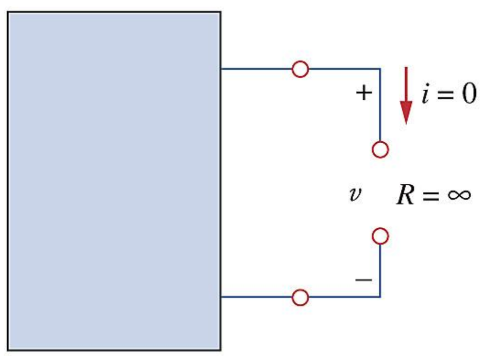
Note

i does not have to be 0.

(ii) When $R = \infty$, we have an **open circuit** and

$$i = \lim_{R \to \infty} \frac{v}{R} = 0$$

indicating that $i = 0$ for any v.

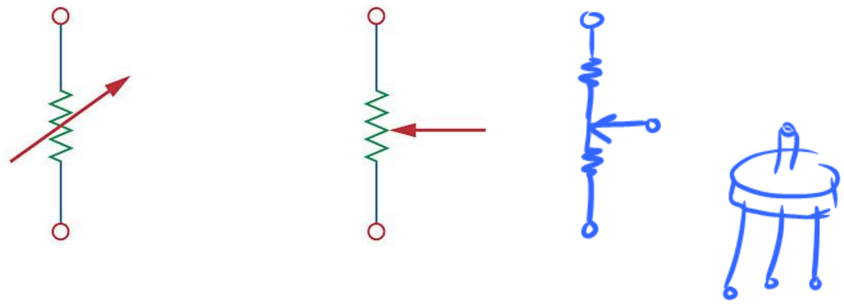


current = 0

v does not have to be 0.

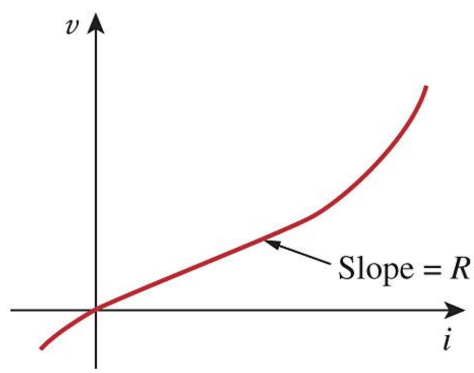
2.1.4. A resistor is either fixed or variable. Most resistors are of the fixed type, meaning their resistance remains constant.

A common variable resistor is known as a **potentiometer** or **pot** for short



2.1.5. Not all resistors obey Ohms law. A resistor that obeys Ohms law is known as a **linear** resistor.

- A **nonlinear** resistor does not obey Ohms law.



- Examples of devices with nonlinear resistance are the lightbulb and the diode.
- Although all practical resistors may exhibit nonlinear behavior under certain conditions, *we will assume in this class that* <u>all elements actually designated as resistors are linear</u>.

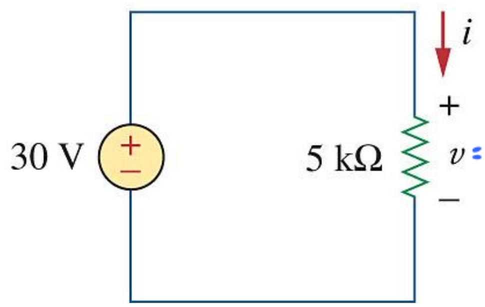
2.1.6. Using Ohm's law, the power p dissipated by a resistor R is

$v = iR$

$$p = vi = i^2 R = \frac{v^2}{R}.$$

$i = \frac{v}{R}$

EXAMPLE 2.1.7. In the circuit below, calculate the current i, and the power p.



$v = 30V$

$v = i \times R$

$30 = i \times 5k$

$i = \frac{30}{5k} = 6mA$

$p = i \times v = 6mA \times 30V$

$= 180\ mW$

DEFINITION 2.1.8. The **power rating** is the maximum allowable power dissipation in the resistor. Exceeding this power rating leads to overheating and can cause the resistor to burn up.

EXAMPLE 2.1.9. Determine the minimum resistor size that can be connected to a 1.5V battery without exceeding the resistor's $\frac{1}{4}$-W power rating.

$\frac{v^2}{R} = P \leq \frac{1}{4}$

$\frac{1.5^2}{R} \leq \frac{1}{4}$

$4 \times 1.5^2 \leq R$

$\parallel$

$\cancel{4} \times \frac{3}{\cancel{2}} \times \frac{3}{\cancel{2}}$

$9\Omega \leq R$

Ohm's law
KVL
KCL
→ Solve almost all circuit problems.
Require a lot of thinking

2.2. Node, Branches and Loops

DEFINITION 2.2.1. Since the elements of an electric circuit can be interconnected in several ways, we need to understand some basic concept of network topology.

- Network = interconnection of elements or devices
- Circuit = a network with closed paths

DEFINITION 2.2.2. **Branch**: A branch represents a single element such as a voltage source or a resistor. A branch represents any two-terminal element.

DEFINITION 2.2.3. **Node**: A node is the point of connection between two or more branches. It is usually indicated by a dot in a circuit.

- If a short circuit (a connecting wire) connects two nodes, the two nodes constitute a single node.

DEFINITION 2.2.4. **Loop**: A loop is any closed path in a circuit. A closed path is formed by starting at a node, passing through a set of nodes and returning to the starting node without passing through any node more than once.

DEFINITION 2.2.5. **Series**: Two or more elements are **in series** if they are cascaded or connected sequentially and consequently carry the *same current*.

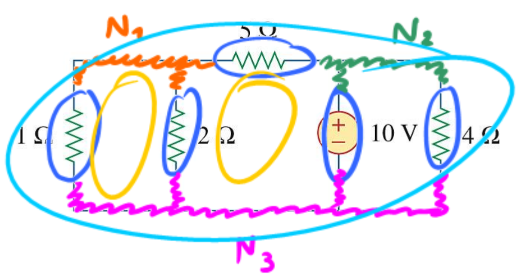
DEFINITION 2.2.6. **Parallel**: Two or more elements are in **parallel** if they are connected to the same two nodes and consequently have the *same voltage* across them.

Elements may be connected in a way that they are neither in series nor in parallel.

EXAMPLE 2.2.7. How many branches and nodes does the circuit in the following figure have? Identify the elements that are in series and in parallel.

5 branches, 3 nodes

$b = \ell + n - 1$
5 3
$\ell = 3$



1Ω // 2Ω
10V // 4Ω
None in series in this configuration

2.2.8. A loop is said to be **independent** if it contains a branch which is not in any other loop. Independent loops or paths result in independent sets of equations. A network with b branches, n nodes, and ℓ independent loops will satisfy the fundamental theorem of network topology:

$$b = \ell + n - 1.$$

← You don't have to remember this formula, but you should know that it exists.

2.3. Kirchhoff's Laws

Ohm's law coupled with Kirchhoff's two laws gives a sufficient, powerful set of tools for analyzing a large variety of electric circuits.

2.3.1. **Kirchhoff's current law (KCL):** the (2) *algebraic sum* of current entering a node (or a closed boundary) is zero. Mathematically,

$$\sum_{n=1}^{N} i_n = 0$$

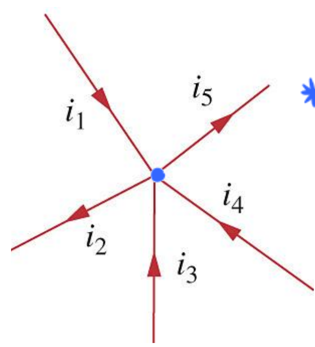
(1) Sum of departing current = 0

*

KCL is based on **the law of conservation of charge**. An alternative form of KCL is

(3) Sum of currents (or charges) entering a node

= Sum of the currents (charges) leaving the node.



KCL:

* (1) $-i_1 + i_2 - i_3 - i_4 + i_5 = 0$

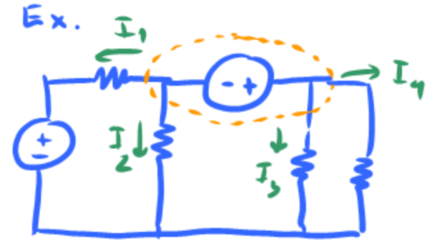
(2) $i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$

(3) $i_1 + i_3 + i_4 = i_2 + i_5$

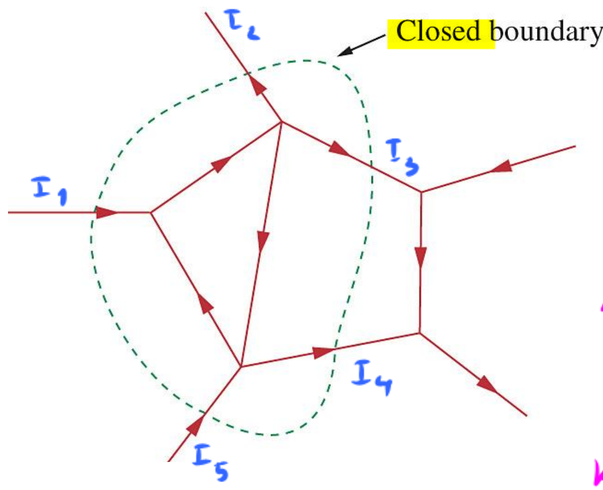
Note that KCL also applies to a closed boundary. This may be regarded as a generalized case, because a node may be regarded as a closed surface shrunk to a point. In two dimensions, a closed boundary is the same as a closed path. The total current entering the closed surface is equal to the total current leaving the surface.

$-I_1+I_2+I_3+I_4-I_5=0$

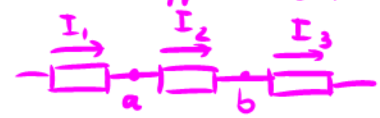
Ex.



$I_1+I_2+I_3+I_4=0$



Another application:

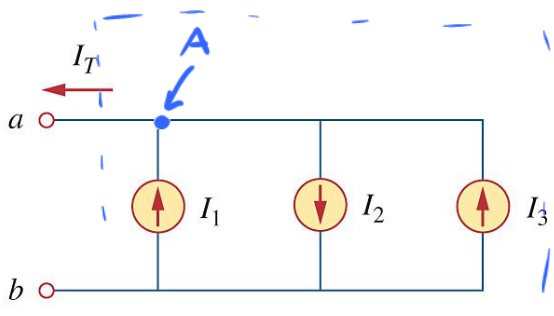


KCL: $I_1=I_2=I_3$

Node a: $-I_1+I_2=0$

$I_1=I_2$

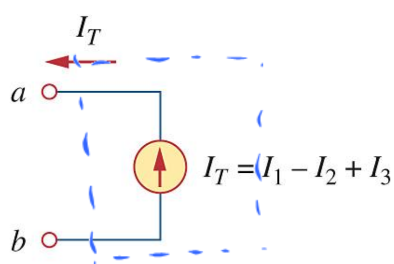
EXAMPLE 2.3.2. A simple application of KCL is combining current sources in parallel.



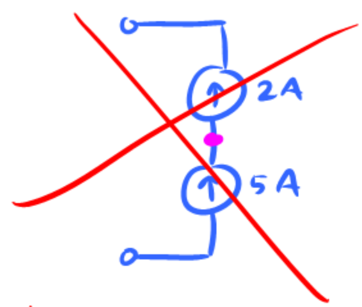
(a)

$I_T-I_1+I_2-I_3=0$

What about current source in series?



(b)



Don't try this.

A Kirchhoff's voltage law (KVL): the algebraic sum of all voltages around a closed path (or loop) is zero. Mathematically,

$$\sum_{m=1}^{M} v_m = 0$$

KVL is based on **the law of conservation of energy**. An alternative form of KVL is

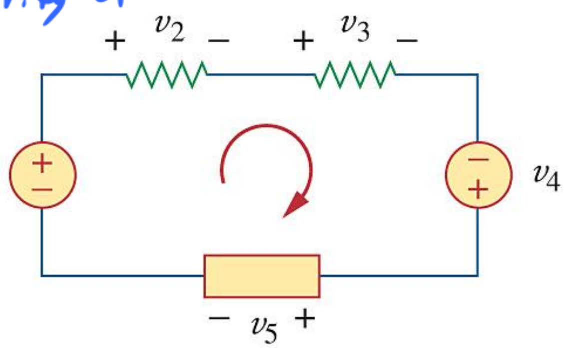
Sum of voltage drops = Sum of voltage rises.

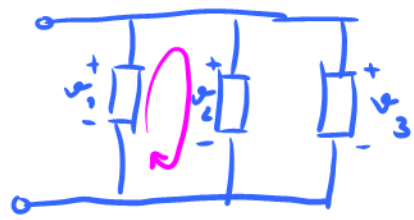
* ① What do I gain/lose

$+v_1 - v_2 - v_3 + v_4 - v_5 = 0$

② The sign is the polarity of the terminal you encantor first

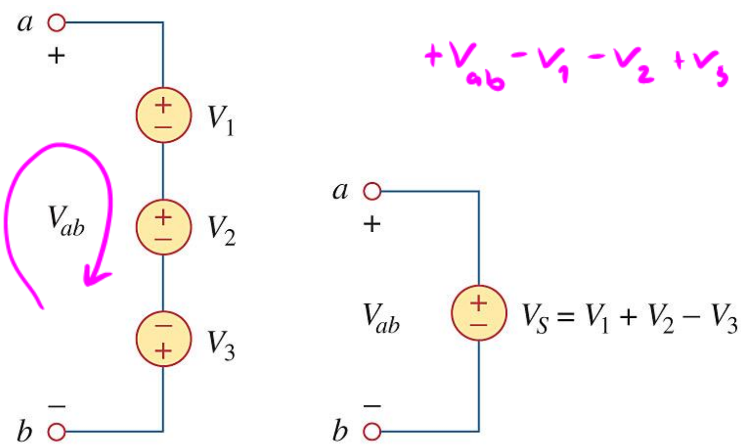
$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$





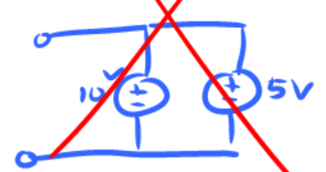
$v_1 = v_2 = v_3$

EXAMPLE 2.3.3. When voltage sources are connected in series, KVL can be applied to obtain the total voltage.



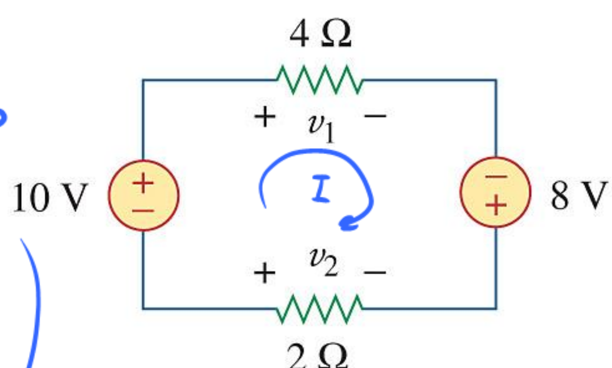
$+V_{ab} - V_1 - V_2 + V_3 = 0$

What about voltage source in parallel?



Don't try this

EXAMPLE 2.3.4. Find v_1 and v_2 in the following circuit.



KVL:

$10 - v_1 + 8 + v_2 = 0$

$v_1 - v_2 = 18$

Ohm's law

$v_1 = I \times 4$

$v_2 = -I \times 2$

$4I - (-2I) = 18$

$6I = 18$

$I = 3$

$v_1 = 3 \times 4 = 12V$

$v_2 = -3 \times 2 = -6V$

2.4. Series Resistors and Voltage Division

2.4.1. When two resistors R_1 and R_2 ohms are connected in series, they can be replaced by an *equivalent* resistor R_{eq} where

$$R_{eq} = R_1 + R_2$$

.

In particular, the two resistors in series shown in the following circuit

same (KCL)

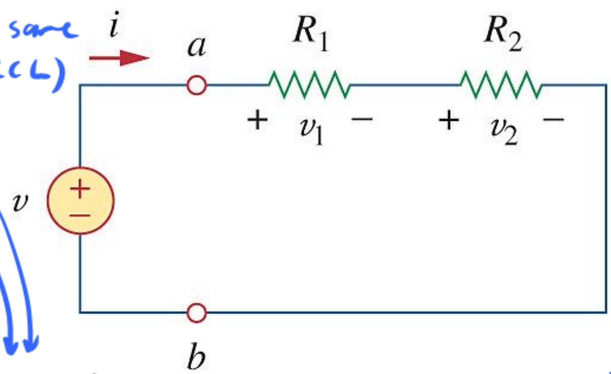
Ohm's law

$v_1 = i_1 R_1 = iR_1$

$v_2 = i_2 R_2 = iR_2$

KVL

$+v - v_1 - v_2 = 0$



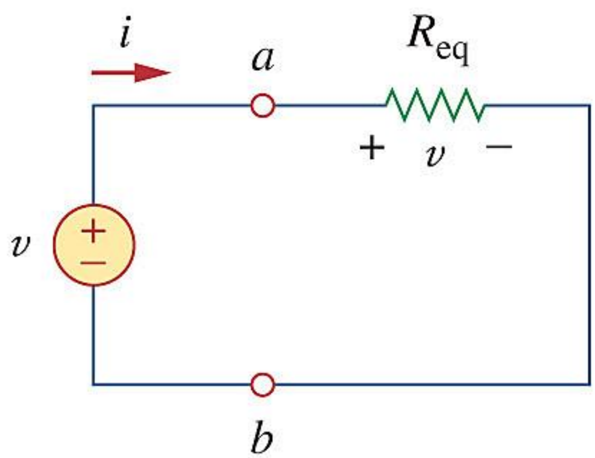
$v_1 = iR_1 = \frac{v}{R_1+R_2} \times R_1 = v\frac{R_1}{R_1+R_2}$

$v_2 = iR_2 = v\frac{R_2}{R_1+R_2}$

$v = v_1 + v_2 = iR_1 + iR_2 = i(R_1+R_2) \leftarrow i = \frac{v}{R_1+R_2}$

can be replaced by an equivalent resistor R_{eq} where $R_{eq} = R_1 + R_2$ as shown below.

$v = i(R_{eq})$



The two circuits above are equivalent in the sense that they exhibit the same voltage-current relationships at the terminals a-b.

Voltage Divider: If R_1 and R_2 are connected in series with a voltage source v volts, the voltage drops across R_1 and R_2 are

$$v_1 = \frac{R_1}{R_1 + R_2} v \qquad \text{and} \qquad v_2 = \frac{R_2}{R_1 + R_2} v$$

Note: The source voltage v is *divided* among the resistors in direct proportion to their resistances.

2.4.2. In general, for N resistors whose values are $R_1, R_2, \ldots, R_N$ ohms connected in series, they can be replaced by an *equivalent* resistor R_{eq} where

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{j=1}^{N} R_j$$

If a circuit has N resistors in series with a voltage source v, the jth resistor R_j has a voltage drop of

$$v_j = \frac{R_j}{R_1 + R_2 + \cdots + R_N} v$$

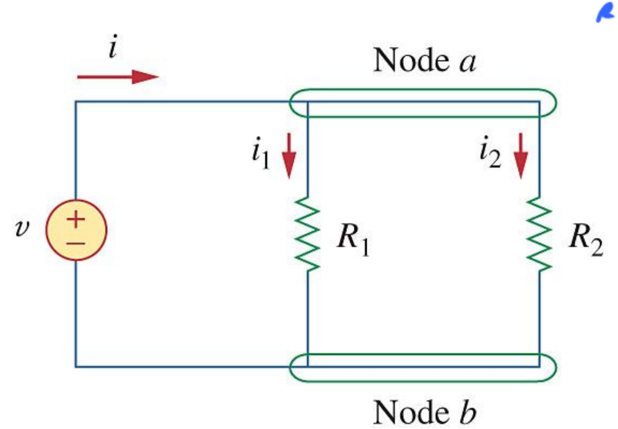
2.5. Parallel Resistors and Current Division

When two resistors R_1 and R_2 ohms are connected in parallel, they can be replaced by an *equivalent* resistor R_{eq} where

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R_{eq} = R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$



Current Divider: If R_1 and R_2 are connected in parallel with a current source i, the current passing through R_1 and R_2 are

$$\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} i = i_1 = \frac{R_2}{R_1 + R_2} i \qquad \text{and} \qquad i_2 = \frac{R_1}{R_1 + R_2} i = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} i$$

Note: The source current i is *divided* among the resistors in inverse proportion to their resistances.

EXAMPLE 2.5.1.

EXAMPLE 2.5.2. $6\|3 =$

EXAMPLE 2.5.3. $(a)\|(na) =$

EXAMPLE 2.5.4. $(ma)\|(na) =$

In general, for N resistors connected in parallel, the *equivalent* resistor $R_{eq} = R_1\|R_2\|\cdots\|R_N$ is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

EXAMPLE 2.5.5. Find R_{eq} for the following circuit.

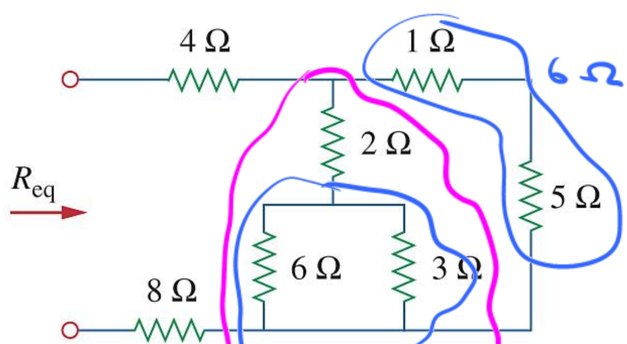
$6/\!/3 = \frac{6\times 3}{6+3} = \frac{18}{9} = 2\ \Omega$

$4 + (6/\!/4) + 8 =$

$4 + 2(3/\!/2) + 8$

1.2

$= 14.4\ \Omega$

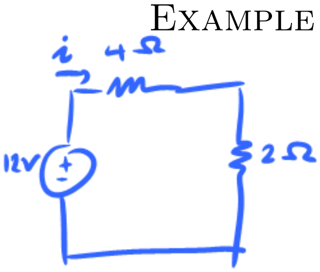


$6\ \Omega$

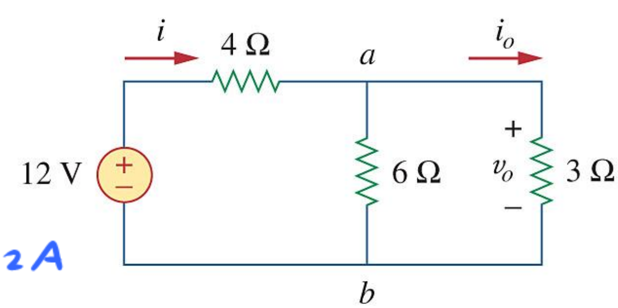
$2\ \Omega$

$2 + 2 = 4\ \Omega$

EXAMPLE 2.5.6. Find i_o, v_o, p_o (power dissipated in the 3Ω resistor).



$i = \frac{12}{4+2} = \frac{12}{6} = 2A$



$i_o = i \times \frac{6}{3+6} = 2 \times \frac{6}{9}$

$= \frac{4}{3} A$

$v_o = i_o R = \frac{4}{3} \times 3 = 4V$

$p_o = \frac{4}{3} \times 4 = \frac{16}{3} W$

EXAMPLE 2.5.7. Three light bulbs are connected to a 9V battery as shown below. Calculate: (a) the total current supplied by the battery, (b) the current through each bulb, (c) the resistance of each bulb.

↳ $R = \frac{V}{I}$

① $P_2 = V_2 I_2 = 15\,W$

$P_3 = V_3 I_2 = 10\,W$

$P_1 = I_1 V_1 = 20\,W$

② KVL (outer loop)

$9 - V_1 = 0 \Rightarrow V_1 = 9V.$

$I_1 = \frac{20}{V_1} = \frac{20}{9}$

③ KVL (left loop)

$9V - V_2 - V_3 = 0$

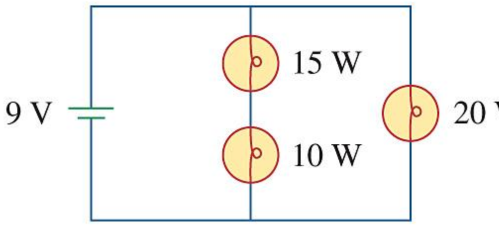
$9 = V_2 + V_3$

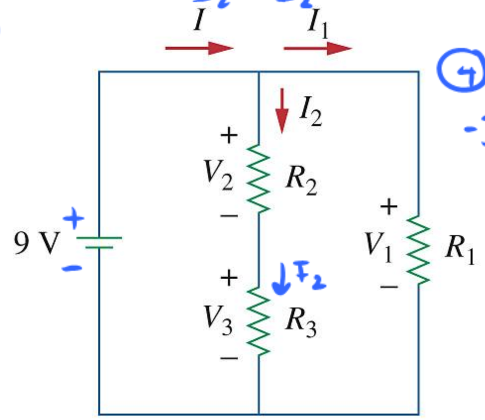
$9 = \frac{15}{I_2} + \frac{10}{I_2} \Rightarrow I_2 = \frac{25}{9}$

④ KCL

$-I + I_1 + I_2 = 0$

$I = I_1 + I_2 = \frac{20}{9} + \frac{25}{9} = \frac{45}{9} = 5\,A$





2.6. Practical Voltage and Current Sources

An ideal voltage source is assumed to supply a constant voltage. This implies that it can supply very large current even when the load resistance is very small.

However, a practical voltage source can supply only a finite amount of current. To reflect this limitation, we model a practical voltage source as an ideal voltage source connected in series with an internal resistance r_s, as follows:

Similarly, a practical current source can be modeled as an ideal current source connected in parallel with an internal resistance r_s.

2.7. Measuring Devices

Ω

Ohmmeter: measures the resistance of the element.
Important rule: Measure the resistance only when the element is disconnected from circuits.

A

Ammeter: connected in **series** with an element to measure current flowing through that element. Since an ideal ammeter should not restrict the flow of current, (i.e., cause a voltage drop), *an ideal ammeter has zero internal resistance.*

V

Voltmeter: connected in **parallel** with an element to measure voltage across that element. Since an ideal voltmeter should not draw current away from the element, *an ideal voltmeter has infinite internal resistance.*

Art $\longrightarrow$ Algorithm

CHAPTER 3

Methods of Analysis

Here we apply the fundamental laws of circuit theory (Ohm's Law & Kirchhoff's Laws) to develop two powerful techniques for circuit analysis.

1. Nodal Analysis (based on KCL)
2. Mesh Analysis (based on KVL)

This is the *most important* chapter for our course.

3.1. Nodal Analysis

Node Voltage.

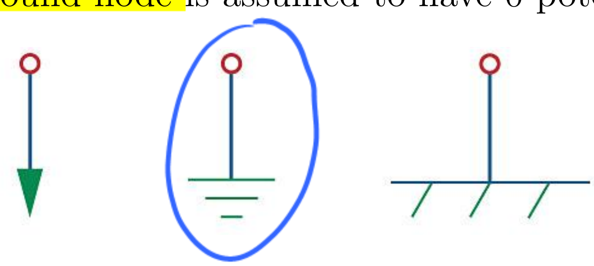
Analyzing circuit using **node voltages** as the circuit variables.

3.1.1. **Steps to Determine Node Voltages:**

Step 0: Determine the number of nodes n.

Step 1: Select a node as a reference node (ground node). Assign voltages v_1, v_2, $\cdots$, v_{n-1} to the remaining $n-1$ nodes. The voltage are referenced with respect to the reference node.

- The ground node is assumed to have 0 potential.



Step 2: Apply KCL to each of the $n-1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.

Step 3: Solve the resulting *simultaneous equations* to obtain the unknown node voltages.

3.1.2. Remark:

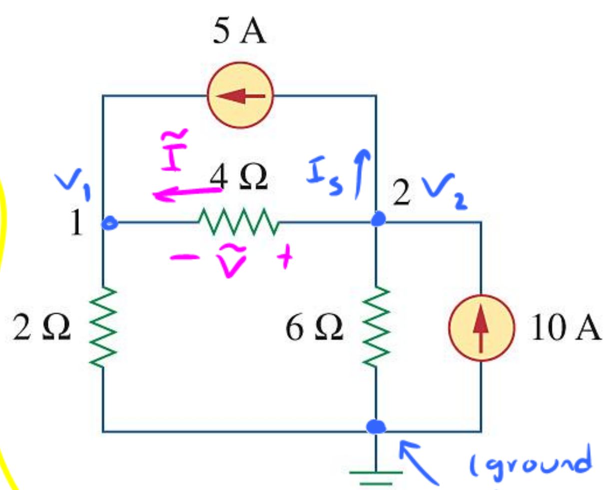
If you are lazy, remember this

$I = \frac{V_a - V_b}{R}$

$\tilde{I} = \frac{V_b - V_a}{R}$

(a) Current flows from a higher potential to a lower potential in a resistor.
(b) If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source.
(c) Multiple methods to solve the simultaneous equations in Step 3.
- Method 1: Elimination technique (good for a few variables)
- Method 2: Write in term of matrix and vectors (write $Ax = b$), then use Cramer's rule.
- Method 3: Use computer (e.g., Matlab) to find A^{-1} and $x = A^{-1}b$

Example 3.1.3. Calculate the node voltages in the circuit below.



$v_o = V_1 - V_2$

$I_o = \frac{V_o}{R} = \frac{V_1 - V_2}{4}$

$\tilde{V} = V_2 - V_1$

$\tilde{I} = \frac{\tilde{V}}{R} = \frac{V_2 - V_1}{4}$

KCL

At node 1,

$-5 + \underbrace{\frac{V_1 - V_2}{4}}_{} + \frac{V_1 - 0}{2} = 0$

current out of node 1 to the right

$V_1 \underbrace{\left(\frac{1}{4} + \frac{1}{2}\right)}_{\frac{3}{4}} - \frac{1}{4} V_2 = 5$

$3V_1 - V_2 = 20 \quad \ldots (1)$

At node 2,

$5 + \frac{V_2 - V_1}{4} - 10 + \frac{V_2}{6} = 0$

$\left(\frac{1}{4} + \frac{1}{6}\right) V_2 - \frac{1}{4} V_1 = 5 \quad \times 12$

$5V_2 - 3V_1 = 60 \quad \ldots (2)$

Method (i) Calculator

Method (ii) (1)+(2)

$-V_2 + 5V_2 = 80$

$4V_2 = 80$

$V_2 = 20\text{ V}$

(1) $V_1 = \frac{20 + V_2}{3} = \frac{40}{3}\text{ V}$

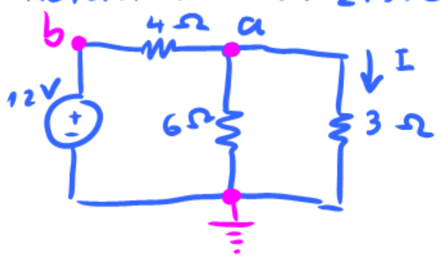
$V_2 = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix}} = \frac{180 - (-60)}{12} = 20$

Method (iii) Cramer's Rule

$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$

$V_1 = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix}} = \frac{100 - (-60)}{15 - 3} = \frac{\cancel{160}\ 40}{\cancel{12}\ 3} = \frac{40}{3}$

Return to Ex. 2.5.6



$V_a = ?$

$I = ?$

$V_b = 12\,V$

KCL at node a: $\frac{V_a-12}{4}+\frac{V_a-0}{3}+\frac{V_a-0}{6}=0$

$V_a\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right)=3$

$\frac{3}{4}V_a = 3$

$V_a = 4\,V$

$I = \frac{4-0}{3} = \frac{4}{3}\,A$

EXAMPLE 3.1.4. Calculate the node voltages in the circuit below.

$i_x = \frac{v_1-v_2}{2}$

KCL

Node 1:

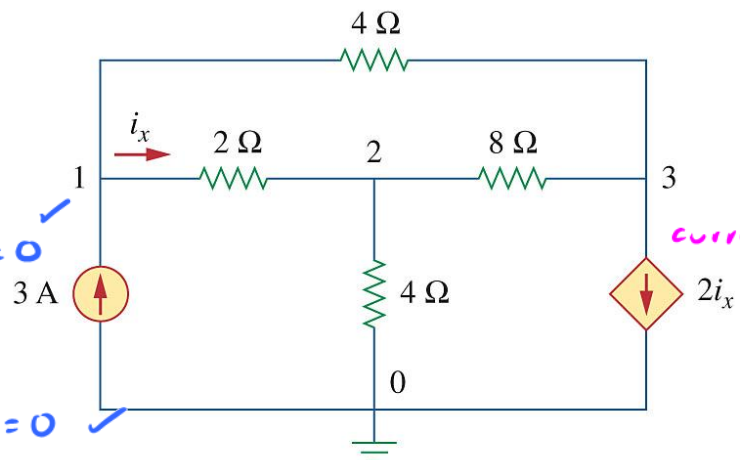
$\frac{v_1-v_3}{4}+\frac{v_1-v_2}{2}+(-3)=0$

Node 2:

$\frac{v_2-v_1}{2}+\frac{v_2-0}{4}+\frac{v_2-v_3}{8}=0$

Node 3:

$\frac{v_3-v_1}{4}+\frac{v_3-v_2}{8}+2\left(\frac{v_1-v_2}{2}\right)=0$



current-controlled current source

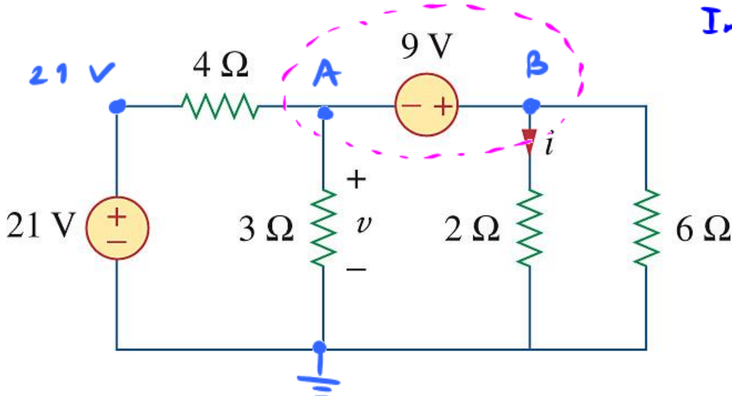
$v_1 = \frac{24}{5} = 4.8\,V$

$v_2 = \frac{12}{5} = 2.4\,V$

$v_3 = -\frac{12}{5} = -2.4\,V$

3.1.5. **Special Case**: If there is a voltage source connected between two nonreference nodes, the two nonreference nodes form a **supernode**. We apply both KCL and KVL to determine the node voltages.

EXAMPLE 3.1.6. Find v and i in the circuit below.



Inside supernode

$V_B - V_A = 9\,V$

At supernode

$\frac{V_A-21}{4}+\frac{V_A-0}{3}+\frac{V_B-0}{2}+\frac{V_B}{6}=0$

$V_A = -0.6\,V$

$V_B = 8.4\,V$

$v = v_A = -0.6\,V$

$i = \frac{V_B-0}{2} = 4.2\,A$

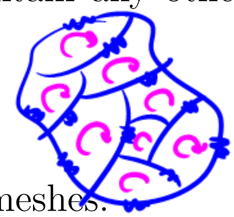
3.1.7. **Remarks**: Note the following properties of a supernode:

(a) The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
(b) A supernode has no voltage of its own.
(c) We can have more than two nodes forming a single supernode.
(d) The supernodes are treated differently because nodal analysis requires knowing the current through each element. However, there is no way of knowing the current through a voltage source in advance.

3.2. Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using **mesh currents** as the circuit variables.

Definition 3.2.1. **Mesh** is a loop which does not contain any other loop within it.



3.2.2. **Steps to Determine Mesh Currents:**

Step 0: Determine the number of meshes n.

Step 1: Assign mesh current $i_1, i_2, \cdots, i_n$, to the n meshes.

- The direction of the mesh current is arbitrary–(clockwise or counterclockwise)–and does not affect the validity of the solution.
- For convenience, we define currents flow in the clockwise (CW) direction.

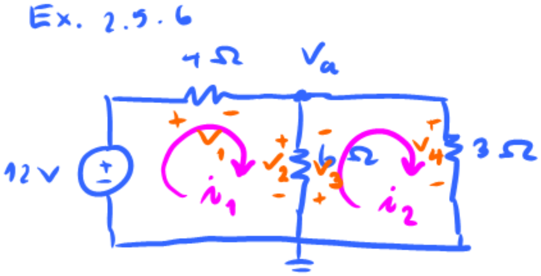
Step 2: From the current direction in each mesh, define the voltage drop polarities.

Step 3: Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh current.

Step 4: Solve the resulting n simultaneous equations to get the mesh current.

$(V_2 = -V_3)$

Ex. 2.5.6



Loop 1:

$+12 - V_1 - V_2 = 0$

$+12 - (i_1 \times 4) - (i_1 - i_2) \times 6 = 0$

Loop 2:

$-V_3 - V_4 = 0$

$-(i_2 - i_1) \times 6 - 3 \times i_2 = 0$

$i_1 = 2A$

$i_2 = \frac{4}{3}A$

$V_a = V_2 = 6 \times (i_1 - i_2)$

$= 6 \times (2 - \frac{4}{3}) = 4V$

Going from a to b, the voltage drops by I×R.

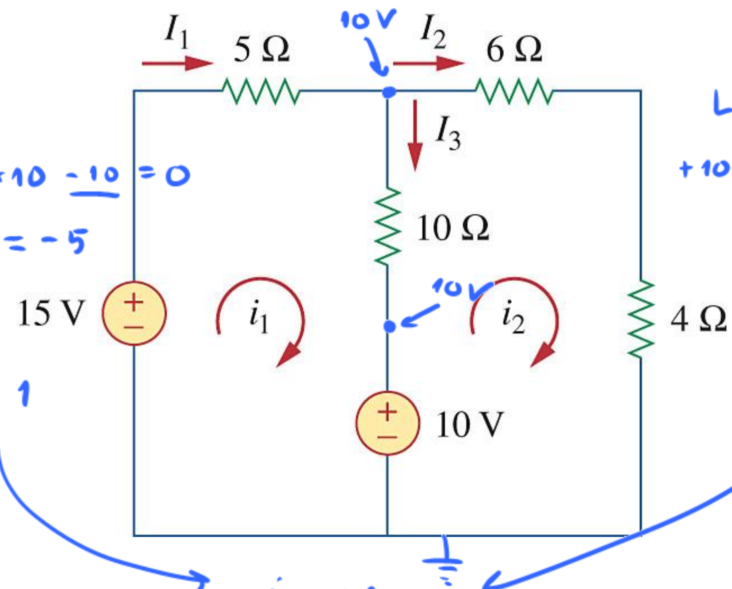
EXAMPLE 3.2.3. Find the branch currents I_1, I_2, and I_3 using mesh analysis.

Loop 1:

$+15 - i_1 \times 5 - (i_1 - i_2)\times 10 - 10 = 0$

$\underbrace{-5 i_1 - i_1 10}_{-15 i_1} + 10 i_2 = -5$

$3 i_1 - 2 i_2 = 1$



Loop 2:

$+10 - 10\times(i_2 - i_1) - 6(i_2) - 4 i_2 = 0$

$i_1 - 2 i_2 = -1$

$i_1 = 1A$

$i_2 = 1A$

$I_1 = i_1 = 1A$

$I_2 = i_2 = 1A$

$I_3 = i_1 - i_2 = 0\ A$

3.2.4. **Remarks**:

(a) Nodal analysis applies KCL to find unknown voltages in a given circuit, while mesh analysis applies KVL to find unknown currents.

(b) Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously.

(c) Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is *planar*.

- A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is nonplanar.

3.3. Nodal Versus Mesh Analysis

You should be familiar with both methods. However, given a network to be analyzed, how do we know which method is better or more efficient?

Suggestion: Choose the method that results in smaller number of variables or equations.

- A circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis.

You can also use one method to check your results of the other method.

Linear System

Input $\xrightarrow{x}$ [S] $\xrightarrow{S(x)}$ Output

Ex.

$x \rightarrow [\ \ ] \rightarrow x^2+1$

$f(x) = x^2+1$

S is a linear system if

① $S(kx) = k\,S(x)$

② $S(x_1+x_2) = S(x_1) + S(x_2)$

Q: Is function $f(x) = x^2+1$ linear?

"affine"

$f(x) = 3x+1$

$f(1) = 4$

$f(2) = 7$

CHAPTER 4

Circuit Theorems

The growth in areas of application of electrical circuits has led to an evolution from simple to complex circuits. To handle such complexity, engineers over the years have developed theorems to simplify circuit analysis. These theorems (Thevenin's and Norton's theorems) are applicable to *linear* circuits which are composed of resistors, voltage and current sources.

4.1. Linearity Property

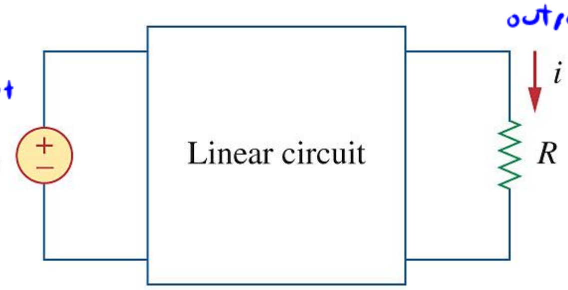
DEFINITION 4.1.1. A **linear circuit** is a circuit whose output is linearly related (or directly proportional) to its input[1]. The input and output can be any voltage or current in the circuit. When we says that the input and output are linearly related, we mean they need to satisfies two properties:

(a) Homogeneous (Scaling): If the input is multiplied by a constant k, then we should observed that the output is also multiplied by k.
(b) Additive: If the inputs are summed then the output are summed.

EXAMPLE 4.1.2. The linear circuit below is excited by a voltage source v_s, which serves as the input.

10 V
1 V
11 V

input



output

2 A
0.2 A
2.2 A

The circuit is terminated by a **load** R. We take the current i through R as the output. Suppose $v_s = 10$ V gives $i = 2$ A. According to the linearity

[1]The input and output are sometimes referred to as cause and effect, respectively.

principle, $v_s = 1V$ will give $i = 0.2$ A. By the same token, $i = 1$ mA must be due to $v_s = 5$ mV.

EXAMPLE 4.1.3. A resistor is a **linear element** when we consider the current i as its input and the voltage v as its output because it has the following properties:

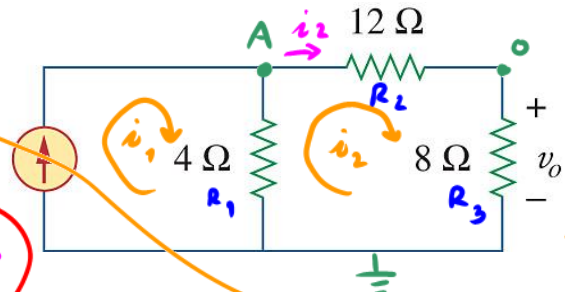
- Homogeneous (Scaling): If i is multiplied by a constant k, then the output v is multiplied by k.

$$iR = v \Rightarrow (ki)R = kv$$

- Additive: If the inputs, i_1 and i_2, are summed then the corresponding output are summed.

$$i_1R = v_1,\ i_2R = v_2 \Rightarrow (i_1 + i_2)R = v_1 + v_2$$

EXAMPLE 4.1.4. For the circuit below, find v_o when (a) $i_s = 15$ and (b) $i_s = 30$.



Method 1: current divider

$i_2 = \frac{R_1}{R_1 + (R_2 + R_3)} \times i_s$

$v_o = i_2 \times R_3 = \frac{R_1 R_3}{R_1 + R_2 + R_3} i_s$

$I_2 = \frac{R_1}{R_1 + R_2} \times I_s$

Method 2: Mesh analysis

$i_1 = i_s$

$-R_1(i_2 - i_s) - i_2 R_2 - i_2 R_3 = 0$

$-R_1 i_2 + R_1 i_s - i_2 R_2 - i_2 R_3 = 0$

$i_2 = \frac{R_1 i_s}{R_1 + R_2 + R_3}$

Method 3
Nodal analysis:

At A: $-i_s + \frac{v_A}{R_1} + \frac{v_A - v_o}{R_2} = 0$

At o: $\frac{v_o - v_A}{R_2} + \frac{v_o}{R_3} = 0$

v_o = same

$v_A = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3} i_s$

Remark: Because $p = i^2R = v^2/R$ (making it a quadratic function rather than a linear one), the relationship between power and voltage (or current) is nonlinear. Therefore, the theorems covered in this chapter are not applicable to power.

Numerical Calculation

$v_o = \frac{4 \times 8}{4 + 12 + 8} i_s = \frac{\cancel{4}^{4} \times \cancel{8}}{\cancel{24}_{\cancel{6}\,3}} i_s$

$= \frac{4}{3} i_s$

When $i_s = 15$ A, $v_o = \frac{4}{\cancel{3}} \times \cancel{15}^{5} = 20$ V

$i_s = 30$ A, $v_o = \frac{4}{\cancel{3}} \times \cancel{30}^{10} = 40$ V

Caution: $i_s = 15A$

$R_3 = 8\Omega \rightarrow v_o = 20$ V

$R_3 = 16\Omega \rightarrow v_o = \frac{4 \times 16}{4 + 12 + 16} \times 15$

$= \frac{{}^{2}4 \times \cancel{16} \times 15}{\cancel{32}}$

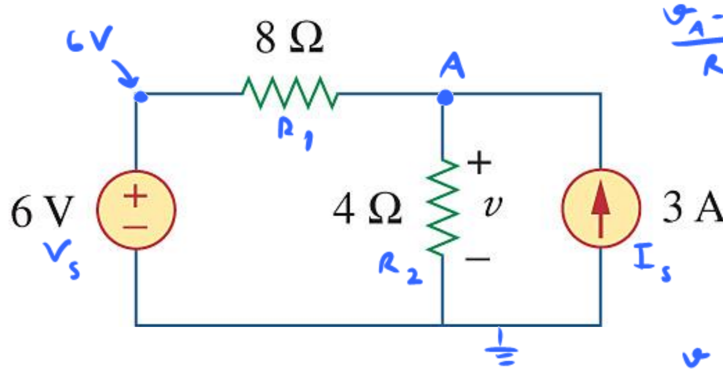
$= 30$ V ✗

4.2. Superposition

Superposition technique = A way to determine currents and voltages in a circuit that has multiple independent sources by considering the contribution of one source at a time and then add them up.

EXAMPLE 4.2.1. Consider the following circuit.

At node A



$$\frac{v_A - V_s}{R_1} + \frac{v_A}{R_2} + (-I_s) = 0$$

$$v\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - \frac{V_s}{R_1} - I_s = 0$$

$$v\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = I_s + \frac{V_s}{R_1}$$

$$v = \frac{R_1 R_2}{R_1+R_2} I_s + \frac{R_2}{R_1+R_2} V_s$$

The **superposition principle** states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

However, to apply the superposition principle, we must keep two things in mind.

1. We consider one independent source at a time while all other independent source are ***turned off***.[2]
   - Replace other independent voltage sources by 0 V (or short circuits)
   - Replace other independent current sources by 0 A (or open circuits)

This way we obtain a simpler and more manageable circuit.

2. Dependent sources are left intact because they are controlled by circuit variable.

---

[2]Other terms such as killed, made inactive, deadened, or set equal to zero are often used to convey the same idea.

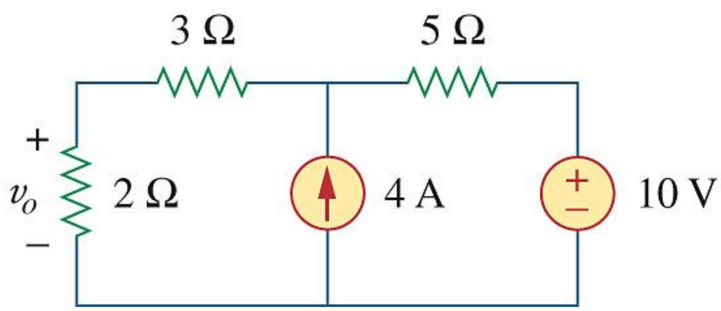
Steps to Apply Superposition Principles:

S1: Turn off all independent sources except one source. Find the output due to that active source.

S2: Repeat S1 for each of the other independent sources.

S3: Find the total contribution by adding algebraically all the contributions due to the independent sources.

EXAMPLE 4.2.2. Using superposition theorem, find v_o in the following circuit.



4.2.3. Keep in mind that superposition is based on linearity. Hence, we cannot find the total power from the power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current and hence it is not linear (e.g. because $5^2 \neq 1^2 + 4^2$).

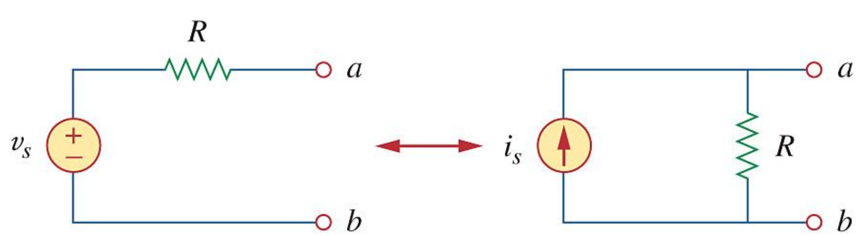
4.2.4. Superposition helps reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

However, it may very likely involve more work. For example, if the circuit has three independent sources, we may have to analyze three circuits. The advantage is that each of the three circuits is considerably easier to analyze than the original one.

4.3. Source Transformation

We have noticed that series-parallel resistance combination helps simplify circuits. The simplification is done by replacing one part of a circuit by its equivalence.[3] Source transformation is another tool for simplifying circuits.

4.3.1. A **source transformation** is the process of replacing a voltage source in series with a resistor R by a current source in parallel with a resistor R or vice versa.

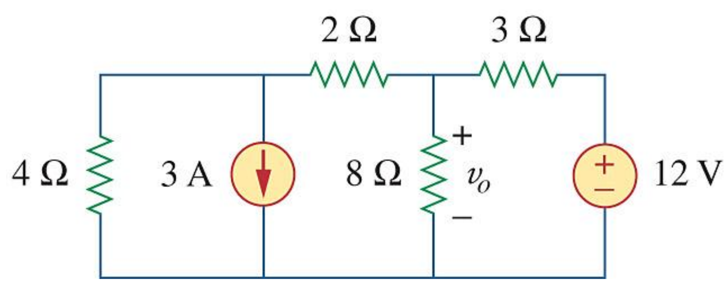


Notice that when terminals $a - b$ are short-circuited, the short-circuit current flowing from a to b is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the righthand side. Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}. \tag{4.1}$$

[3]Recall that an equivalent circuit is one whose $v - i$ characteristics are identical with the original circuit.

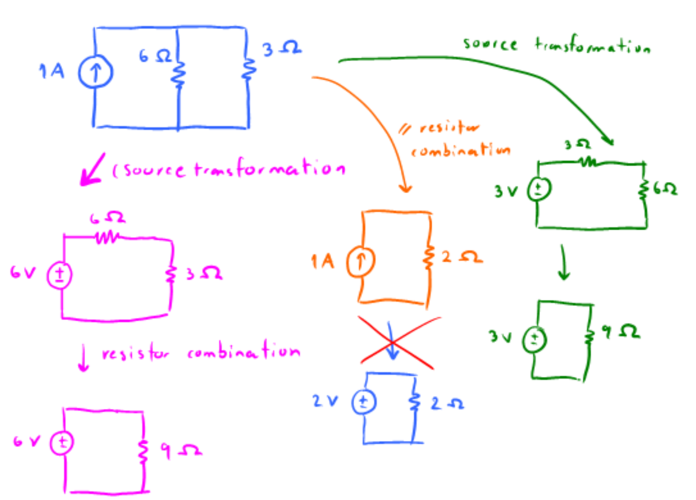
EXAMPLE 4.3.2. Use source transformation to find v_0 in the following circuit:



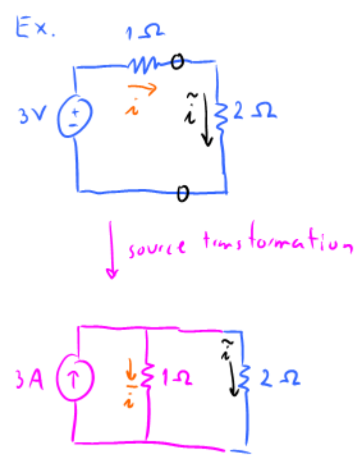
Remark: From (4.1), an ideal voltage source with $R = 0$ cannot be replaced by a finite current source. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.

Caution: Do not include the element whose voltage/current are to be determined.

Ex. Consider this circuit



Ex.



$\tilde{i} = \frac{3}{1+2} = \frac{3}{3} = 1\,A$ ✓

$i = \tilde{i} = 1\,A$

$\neq$

$\tilde{i} = \frac{1}{1+2} \times 3 = \frac{3}{3} = 1\,A$ ✓

$i = 2\,A$

4.4. Thevenin's Theorem

It often occurs in practice that a particular element in a circuit is variable (usually called the **load**) while other elements are fixed.

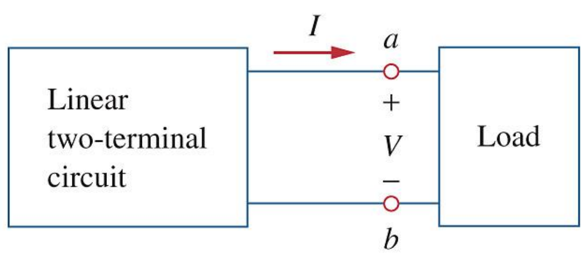
- As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load.

Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenins theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

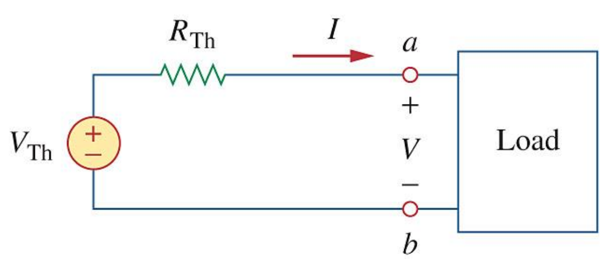
Thevenin's Theorem is an important method to simplify a complicated circuit to a very simple circuit. It states that a circuit can be replaced by an equivalent circuit consisting of an independent voltage source V_{Th} in series with a resistor R_{Th}, where

V_{Th}: the open circuit voltage at the terminal.

R_{Th}: the equivalent resistance at the terminals when the independent sources are turned off.



(a)

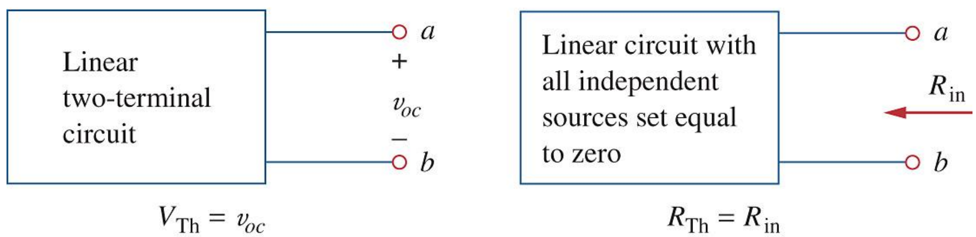


(b)

4.4.1. **Steps to Apply Thevenin's theorem. (Case I: No dependent source)**

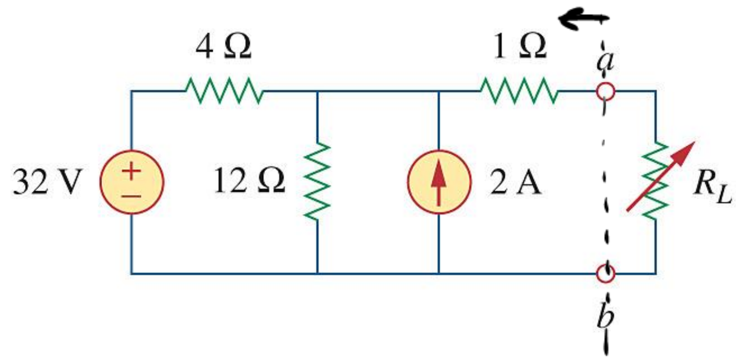
S1: Find R_{Th}: Turn off all independent sources. R_{TH} is the input resistance of the network looking between terminals a and b.

S2: Find V_{Th}: Open the two terminals (remove the load) which you want to find the Thevenin equivalent circuit. V_{Th} is the open-circuit voltage across the terminals.

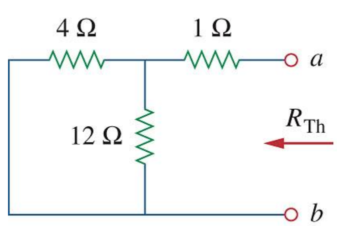


S3: Connect V_{Th} and R_{Th} in series to produce the Thevenin equivalent circuit for the original circuit.

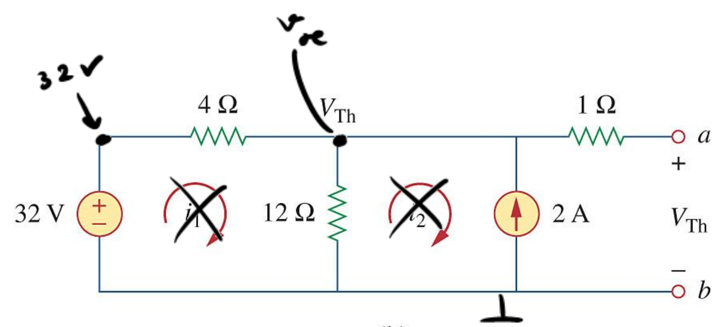
EXAMPLE 4.4.2. Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals a-b. Then find the current through $R_L = 6, 16,$ and 36Ω.



Solution:



(a)



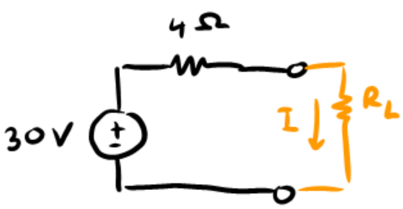
(b)

$$R_{Th} = 1 + (12 /\!/ 4) = 1 + 3 = 4\ \Omega$$

Nodal analysis

$$\frac{v_a - 32}{4} + \frac{v_a}{12} - 2\text{A} = 0$$

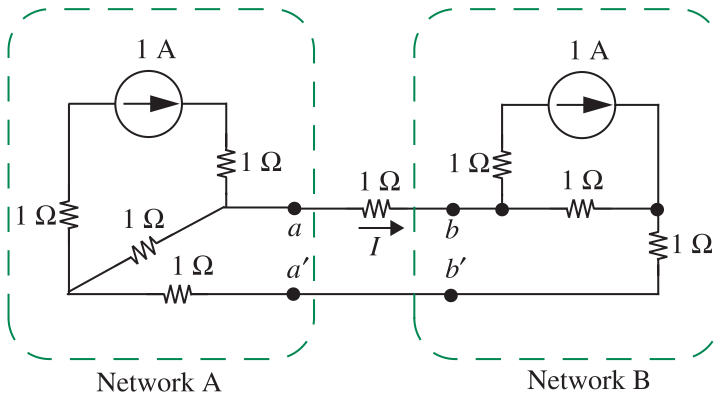
$$\Rightarrow v_a = 30\text{V}$$



$$I = \frac{30}{4 + R_L}$$

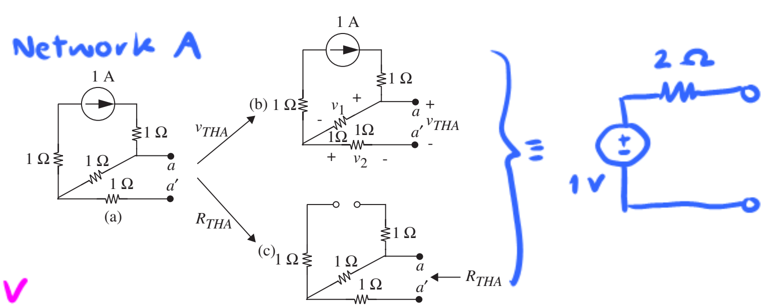
R_L	I
6	3A
16	1.5A
36	0.75A

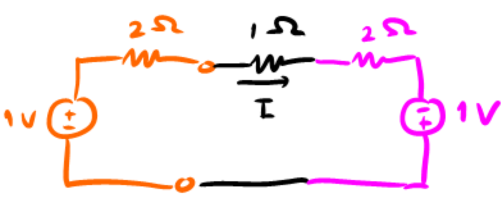
EXAMPLE 4.4.3. Determine the current I in the branch ab in the circuit below.



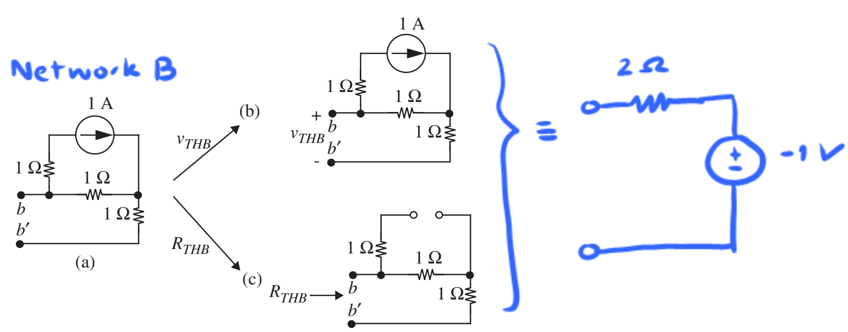
Solution:

There are many approaches that we can take to obtain the current I. For example, we could apply the node method and determine the node voltages at nodes a and b and thereby determine the current I. However, we will find the Thvenin equivalent network for the subcircuit to the left of the aa' terminal pair (Network A) and for the subcircuit to the right of the bb' terminal pair (Network B), and then using these subcircuits solve for the current I.





$I = \frac{1+1}{2+1+2} = \frac{2}{5}$ A



4.4.4. **Steps to Apply Thevenin's theorem.(Case II: with dependent sources)**

S1: Find R_{TH}:

S1.1 Turn off all independent sources (but leave the dependent sources on).

S1.2 Apply a voltage source v_o at terminals a and b, determine the resulting current i_o, then

$$R_{TH} = \frac{v_o}{i_o}$$

Note that: We usually set $v_o = 1$ V.

Or, equivalently,

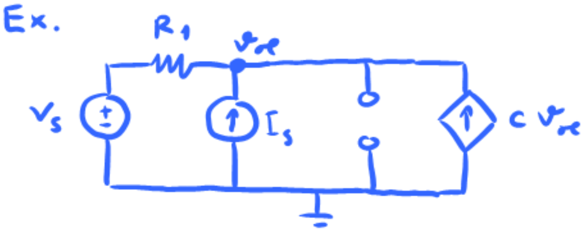
S1.2′ Apply a current source i_o at terminal a and b, find v_o, then $R_{TH} = \frac{v_o}{i_o}$

S2: Find V_{TH}, as the open-circuit voltage across the terminals.

S3: Connect R_{TH} and V_{TH} in series.

Remark: It often occurs that R_{TH} takes a negative value. In this case, the negative resistance implies that the circuit is supplying power. This is possible in a circuit with dependent sources.

Ex.

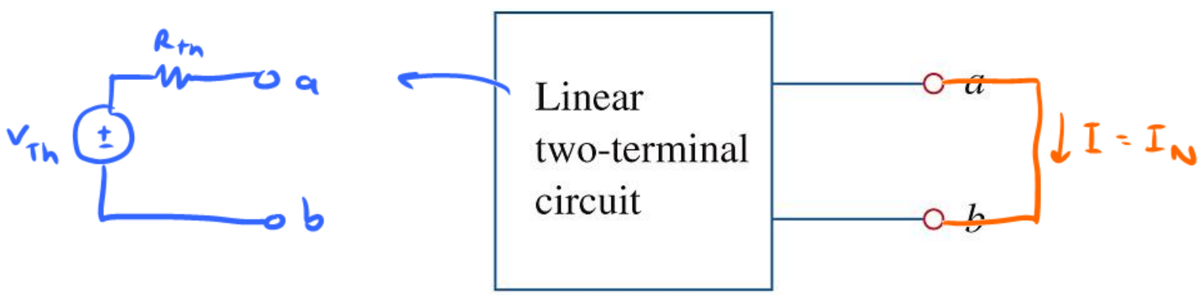


4.5. Norton's Theorem

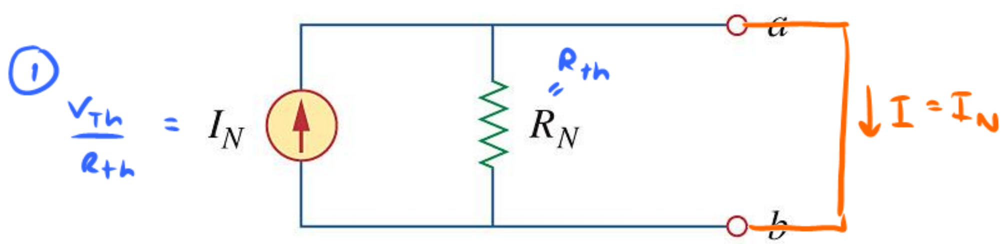
Norton's Theorem gives an alternative equivalent circuit to Thevenin's Theorem.

Norton's Theorem: A circuit can be replaced by an equivalent circuit consisting of a **current source** I_N in **parallel** with a **resistor** R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Note: $R_N = R_{TH}$ and $I_N = \frac{V_{TH}}{R_{TH}}$. These relations are easily seen via source transformation.[4]



(a)



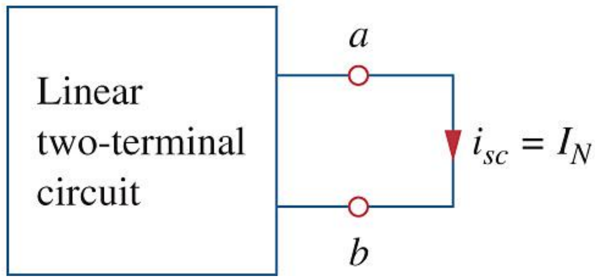
(b)

[4]For this reason, source transformation is often called Thevenin-Norton transformation.

Steps to Apply Norton's Theorem

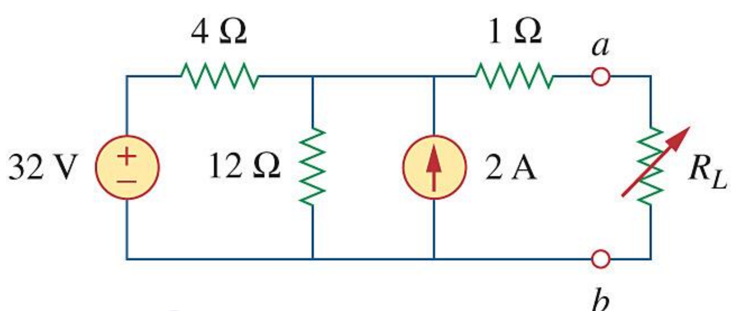
S1: Find R_N (in the same way we find R_{TH}).

S2: Find I_N: Short circuit terminals a to b. I_N is the current passing through a and b.

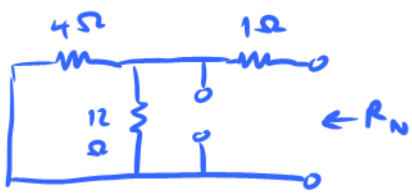


S3: Connect I_N and R_N in parallel.

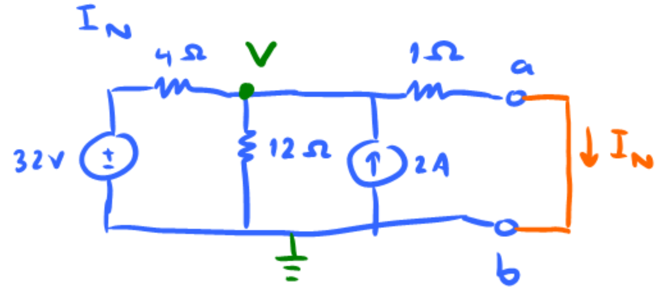
Example 4.5.1. Find the Norton equivalent circuit of the circuit shown below, to the left of the terminals a-b.



$R_N = 1 + (4 /\!/ 12) = 1 + 3 = 4\,\Omega$



I_N

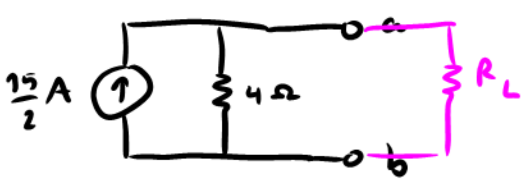


Nodal Analysis:

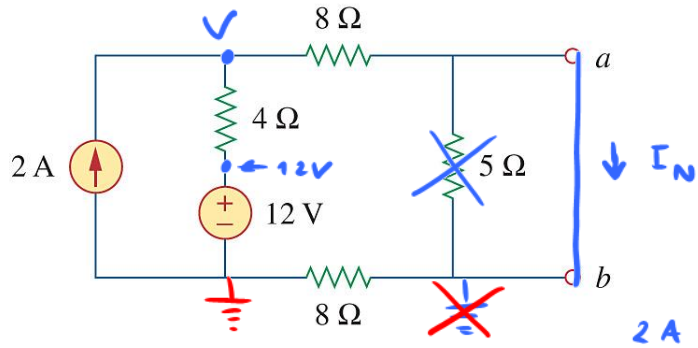
$$\frac{V-32}{4} + \frac{V}{12} + (-2) + \frac{V-0}{1} = 0$$

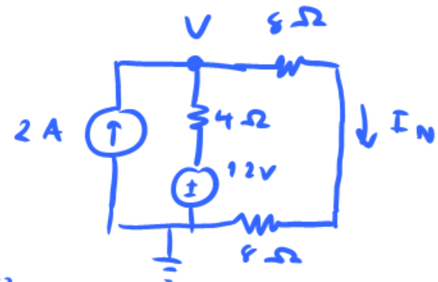
$$\Rightarrow V = \frac{15}{2}\,V$$

$$I_N = \frac{V}{1\Omega} = \frac{15}{2}\,A$$

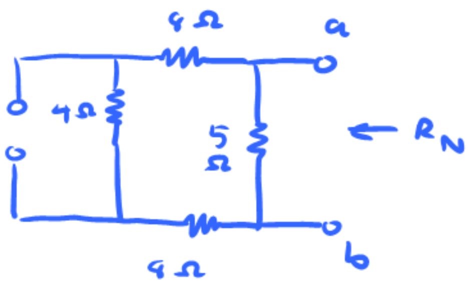


EXAMPLE 4.5.2. Find the Norton equivalent circuit of the circuit in the following figure at terminals a-b.





R_N

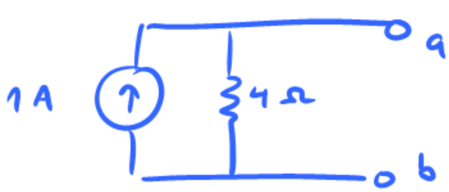


$$-2 + \frac{V-12}{4} + \frac{V-0}{16} = 0$$

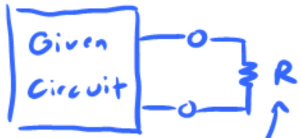
$$\Rightarrow V = 16\,V$$

$$I_N = \frac{V-0}{16} = \frac{16}{16} = 1\,A$$

$R_N = 20 // 5 = 4\,\Omega$



Goal: Find the value of R such that maximum power gets delivered to R. (transferred)



Source

The load in this case is a resistor.

4.6. Maximum Power Transfer

In many practical situations, a circuit is designed to provide power to a load. In areas such as communications, it is desirable to maximize the power delivered to a load. We now address the problem of delivering the maximum power to a load when given a system with known internal losses.

The Thevenin and Norton models imply that some of the power generated by the source will necessarily be dissipated by the internal circuits within the source.

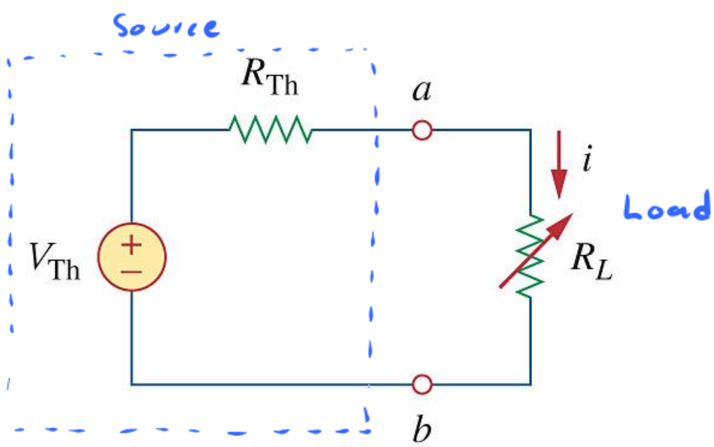
Questions:

(a) How much power can be transferred to the load under the most ideal conditions?

(b) What is the value of the load resistance that will absorb the maximum power from the source?

If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown below, the power delivered to the load resistor R_L is

$$p = i^2 R_L \quad \text{where} \quad i = \frac{V_{th}}{R_{th} + R_L}$$



The derivative of p with respect to R_L is given by

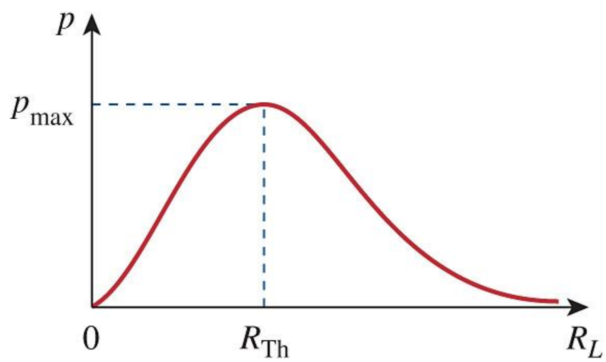
$$\begin{aligned}
\frac{dp}{dR_L} &= 2i\frac{di}{dR_L}R_L + i^2 \\
&= 2\frac{V_{th}}{R_{th}+R_L}\left(-\frac{V_{th}}{(R_{th}+R_L)^2}\right) + \left(\frac{V_{th}}{R_{th}+R_L}\right)^2 \\
&= \left(\frac{V_{th}}{R_{th}+R_L}\right)^2\left(-\frac{2R_L}{R_{th}+R_L}+1\right).
\end{aligned}$$

We then set this derivative equal to zero and get

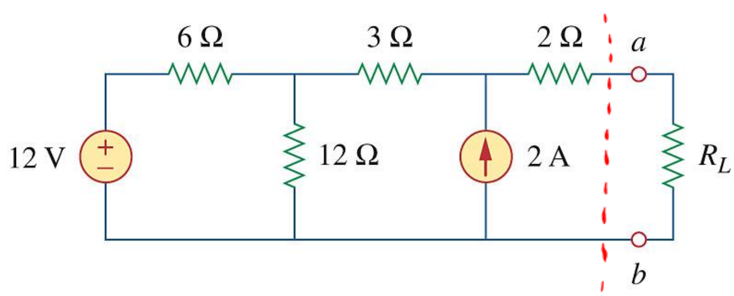
$$R_L = R_{TH}.$$

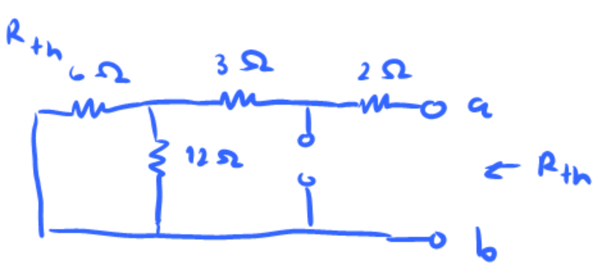
4.6.1. The maximum power transfer takes place when the load resistance R_L equals the Thevenin resistance R_{Th}. As a consequence, the maximum power transferred to R_L equals to

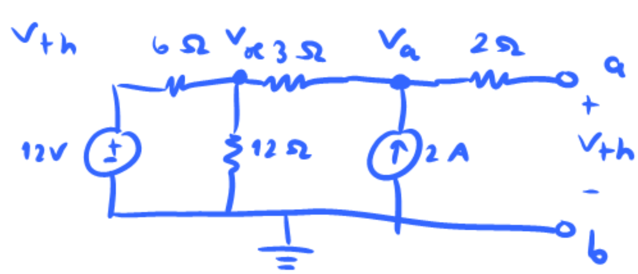
$$p_{\max} = \left(\frac{V_{th}}{R_{th} + R_{th}}\right)^2 R_{th} = \boxed{\frac{V_{th}^2}{4R_{th}}}.$$



EXAMPLE 4.6.2. Find the value of R_L for maximum power transfer in the circuit below. Find the maximum power.







$R_{th} = \underbrace{(6/\!/12)}_{4} + 3 + 2 = 4 + 5 = 9$

CHAPTER 5

Operational Amplifiers

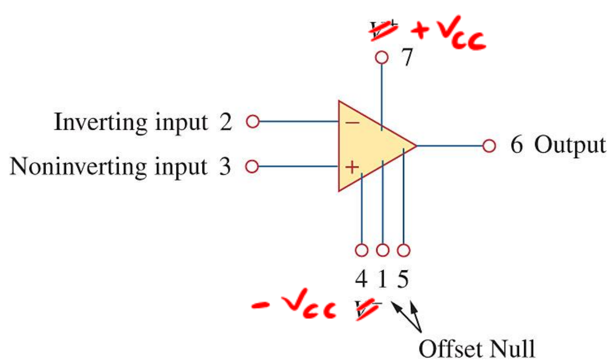
Operational amplifiers (or **Op Amp**) is an active circuit element that can perform mathematical operations between signals (e.g., amplify, sum, subtract, multiply, divide, integrate, differentiate).

- The ability of the op amp to perform these mathematical operations is the reason it is called an operational amplifier. It is also the reason for the widespread use of op amps in analog design.

Op amp is a building block of modern electronic instrumentation. Therefore, mastery of operational amplifier fundamentals is paramount to any practical application of electronic circuits. They are popular in practical circuit degigns because they are versatile, inexpensive, easy to use, and fun to work with.

An op amp consisting of a complex arrangement of resistors, transistors, capacitors, and diodes. Here, we ignore the details.

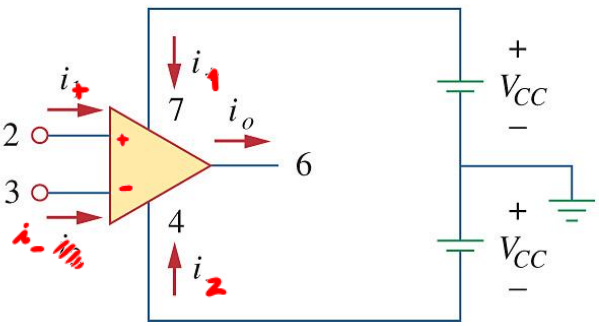
The circuit symbol for the op amp is shown below.



- It has two inputs and one output.
- The inputs are marked with minus (-) and plus (+) to specify inverting and noninverting inputs, respectively.

- An input applied to the noninverting terminal will appear with the same polarity at the output, while an input applied to the inverting terminal will appear inverted at the output.

As an active element, the op amp must be powered by a voltage supply:

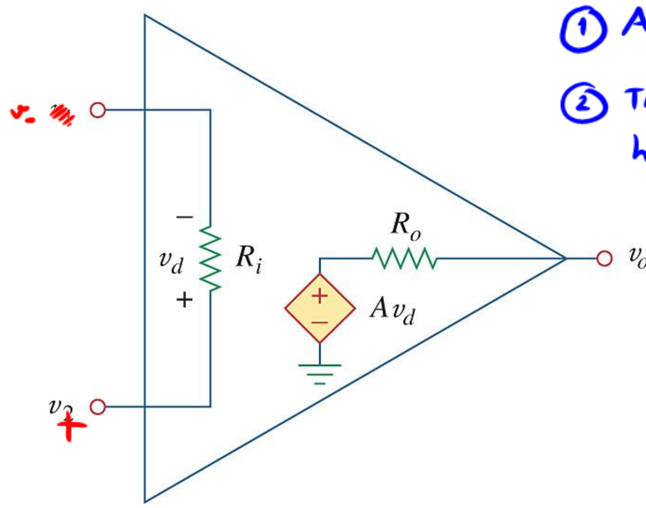


Although the power supplies are often ignored in op amp circuit diagrams for the sake of simplicity, the power supply currents must not be overlooked:

$$i_o = i_1 + i_2 + i_+ + i_-.$$

The equivalent circuit of non-ideal op amp is shown below. Note that the output section consists of a voltage-controlled source in series with the output resistance R_o.

Remember
1. All ⏚ are short together.
2. There is always one hidden ⏚ at V_{cc}.



Input-output relations:

$$v_0 = Av_d = \cancel{A(v_2 - v_1)}$$
$$= A(v_+ - v_-)$$

where

$$v_0 = \text{voltage between the output terminal and ground}$$
$$v_1 = \text{voltage between the inverting terminal and ground}$$
$$v_2 = \text{voltage between the noninverting terminal and ground}$$
$$v_d = v_2 - v_1 = \text{differential input voltage}$$
$$A = \text{open-loop voltage gain}$$

- In words, the op amp senses the difference between the two inputs, multiplies it by the gain A, and causes the resulting voltage to appear at the output.

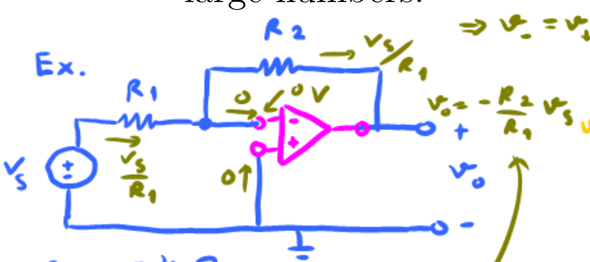
A is called the open-loop voltage gain because it is the gain of the op amp ***without any external feedback*** from output to input. Typical ranges for op amp parameters are shown in the following table Working

Parameter	Typical range	Ideal values
Open-loop gain, A	10^5 to 10^8	∞
Input resistance, R_i	10^5 to 10^{13} Ω	∞ Ω
Output resistance, R_o	10 to 100 Ω	0 Ω
Supply voltage, V_{cc}	5 to 24 V	

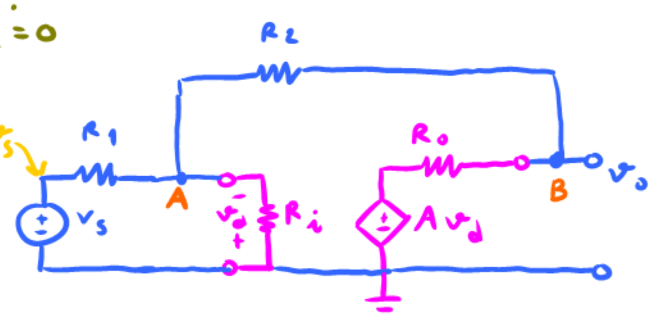
with a nonideal op amp is tedious because it involves dealing with very large numbers. Two rules:

$\Rightarrow v_- = v_+ = 0$

Ex.



$v_o = -\frac{R_2}{R_1} v_s$



$R_1 = 10\ \text{k}\Omega$

$R_2 = 20\ \text{k}\Omega$

$A = 200{,}000$
$R_i = 2\ \text{M}\Omega$
$R_o = 50\ \Omega$ $\Big\}$ inside op amp

$\frac{v_o}{v_s} = -\frac{R_2}{R_1}$

Find $\frac{v_o}{v_s}$

This is the close-loop gain because R_2 feedback resistor closes the loop between the output and input terminals.

At node A,

$$\frac{v_A - v_s}{R_1} + \frac{v_A}{R_i} + \frac{v_A - v_B}{R_2} = 0$$

At node B,

$$\frac{v_B - v_A}{R_2} + \frac{v_B - A v_d}{R_o} = 0$$

Note that $v_d = -v_A$

$$v_B = -\frac{15999999800}{8000120601} v_s$$
$$\approx -1.99996982520459\, v_s$$
$$\approx -2 v_s$$

Note that $v_B = v_o$

So

$$v_o \approx -2 v_s$$

$$\frac{v_o}{v_s} \approx -2$$

Working with a nonideal op amp is tedious because you have to deal with very large number.

Using this model, you will get $\frac{v_o}{v_s} \approx -\frac{R_2}{R_1}$ easily.

5.1. Ideal Op-Amp

To facilitate understanding, we assume ideal op amps with the ideal values above.

Definition 5.1.1. An ***ideal op amp*** is an amplifier with infinite open-loop gain (A), infinite input resistance (R_i), and ~~zer o~~ 0 output resistance (R_o).

Unless stated otherwise, we will assume from now on that every op amp is ideal.

<u>**Two important characteristics** of the ideal op-amp:</u>

(a) The current into both input terminals are zero.

$$i_- \cancel{i_1} = 0, \quad i_+ \cancel{i_2} = 0.$$

(b) The voltage across the input terminals is negligibly small.

$$\cancel{v_d = v_2 - v_1 \approx 0}$$

$$v_d = v_+ - v_- \approx 0$$

or

$$\cancel{v_1 = v_2}.$$

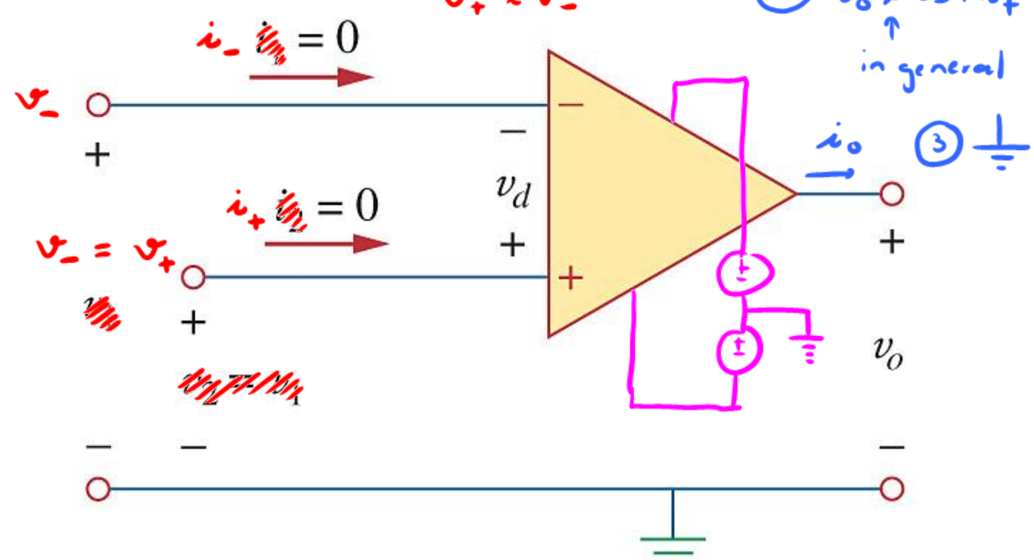
$$v_+ \approx v_-$$

Caution

① $v_- = v_+$ DOES NOT mean $= 0$.

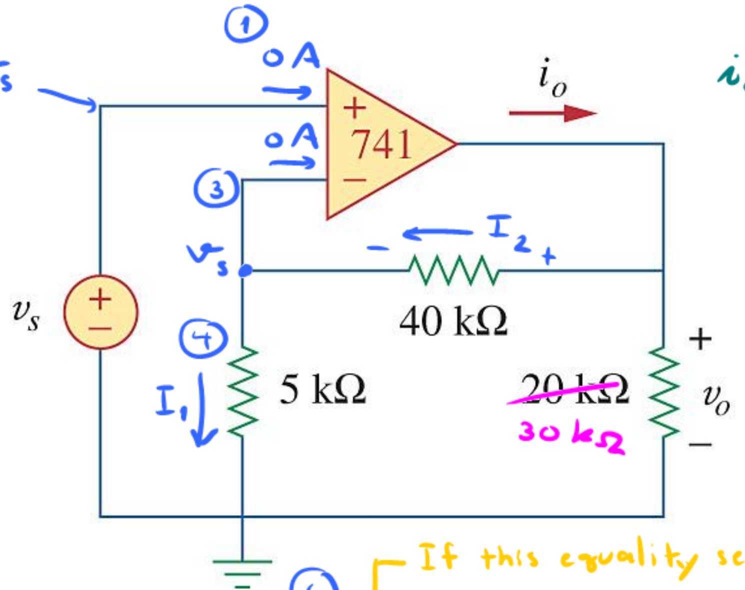
② $i_o \neq i_- + i_+$ in general

③ ⏚



Ex.(ideal op amp with feedback) An ideal op amp is used in the circuit below. Find the closed-loop gain[1] v_o/v_s. Determine current i_o when $v_s = 1$ V.

② $v_+ = v_s$



$i_o = I_2 + \frac{v_o}{20k\Omega}$

$= \frac{v_s}{5k} + \frac{9v_s}{20k}$

$= \frac{1}{5k} + \frac{9}{20k}$

$= \frac{20+45}{100k} = \frac{65}{100k}$

$= 0.65$ mA

③ $v_- = v_+ = v_s$

④ $I_1 = \frac{v_s}{5k}$

⑤ $I_2 = I_1$ (because $i_- = 0$)

⑥ $v_o = v_s + I_2 \times 40k$

$= v_s + \frac{v_s}{\cancel{5k}} \times \cancel{40k}^{\,8}$

$= 9v_s$

$\frac{v_o}{v_s} = 9$

If this equality seem confusing, it may be easier to start with

$I_2 = \frac{v_o - v_s}{40k}$

Q: Will my $\frac{v_o}{v_s}$ change if 20 kΩ is replaced by 30 kΩ?

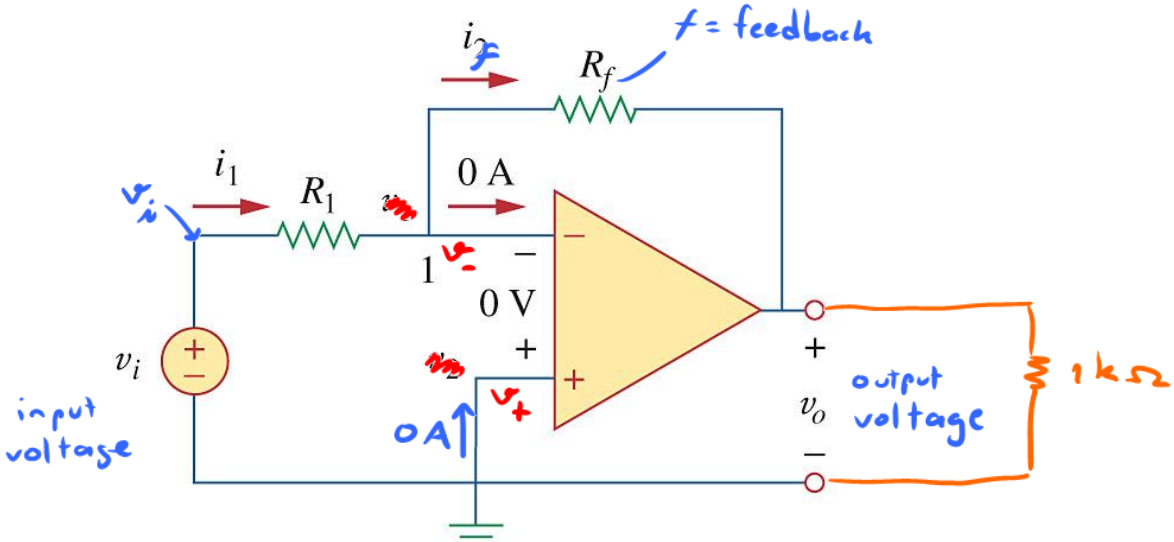
A: No.

[1]**closed-loop gain** = ratio of the output voltage to the input voltage.

5.2. Inverting Amplifier

Op amp can be used in circuits as modules for creating more complex circuits. The first of such op-amp circuits is the inverting amplifier which **reverses the polarity of the input signal while amplifying it**.

Let's say we want to amplify a small signal v_i by a factor of A_v and reverse its polarity as well, i.e., the output signal v_o is $v_o = -A_v v_i$. We can achieve this using an inverting amplifier. A key feature of the inverting amplifier is that both the input signal and the feedback are applied at the inverting terminal of the op amp.



① $i_- = i_+ = 0$ A

$v_+ = v_-$

② $v_+ = 0$ V

③ $v_- = 0$ V

④ $i_1 = \dfrac{v_i - 0}{R_1} = \dfrac{v_i}{R_1}$

⑤ $i_f = i_1$

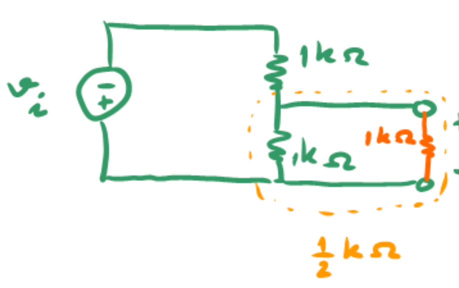
$i_f = \dfrac{0 - v_o}{R_f}$

⑥ $v_o = 0 - i_f R_f$

$= -\dfrac{v_i}{R_1} \times R_f = -\dfrac{R_f}{R_1} \times v_i$

Example:
Suppose
$R_f = 1\,k\Omega$
$R_1 = 2\,k\Omega$
Then
$v_o = -\frac{1}{2} v_i$

Consider

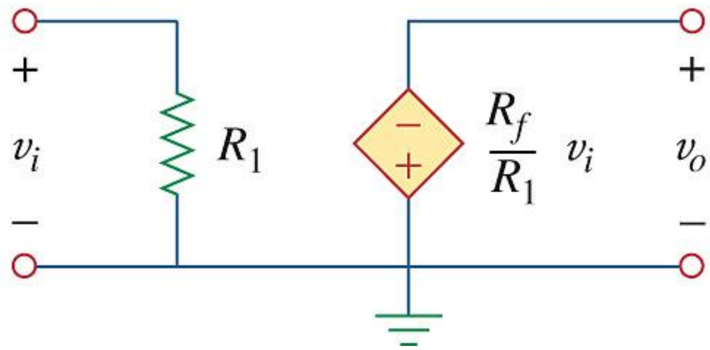


if $v_i = 9$ — w/o extra 1kΩ, $v_o = -\frac{9}{2} = -4.5$ V; w/ extra 1kΩ, $v_o = -3$ V

$v_o = -\frac{1}{2} v_i$

$v_o = -\dfrac{\frac{1}{2}k}{\frac{1}{2}k + 1k} \times 9 = -\dfrac{1}{1+2} \times 9 = -3$ V

The equivalent circuit for the inverting amplifier is:

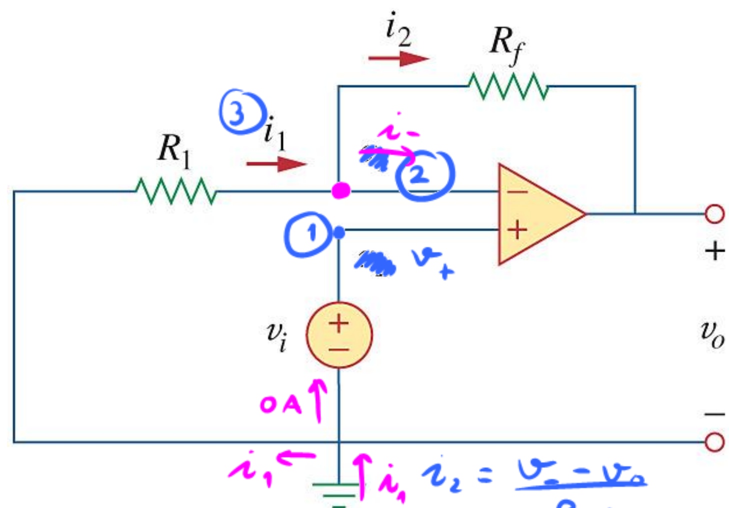


The voltage gain is $A_v = v_o/v_i = -R_f/R_1$.

5.3. Noninverting Amplifier

A noninverting amplifier amplifies a signal by a constant positive gain (no inversion of polarity). The circuit for a noninverting amplifier is

$-i_1 + i_2 + \cancel{i_-} = 0$



$i_2 = \frac{v_- - v_o}{R_f}$

① $v_+ = v_i$

② $v_- = v_+ = v_i$

③ $i_1 = \frac{0 - v_-}{R_1} = -\frac{v_i}{R_1}$

④ $i_- = 0$

⑤ $i_2 = i_1 = -\frac{v_i}{R_1}$

⑥ $v_o = v_- - i_2 R_f$

$= v_i - \left(-\frac{v_i}{R_1}\right) R_f$

$= v_i + \frac{R_f}{R_1} v_i$

$= v_i \left(1 + \frac{R_f}{R_1}\right)$

$$\frac{v_o}{v_i} = 1 + \frac{R_f}{R_1}$$

Special cases

① $R_f = 0$

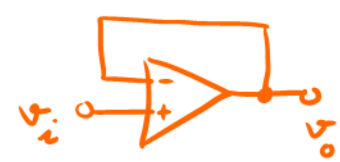
or

② $R_1 = \infty$

or both

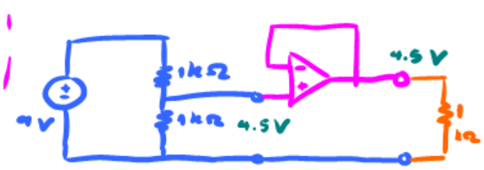
$\frac{v_o}{v_i} = 1 + 0 = 1$

$\Rightarrow v_o = v_i$



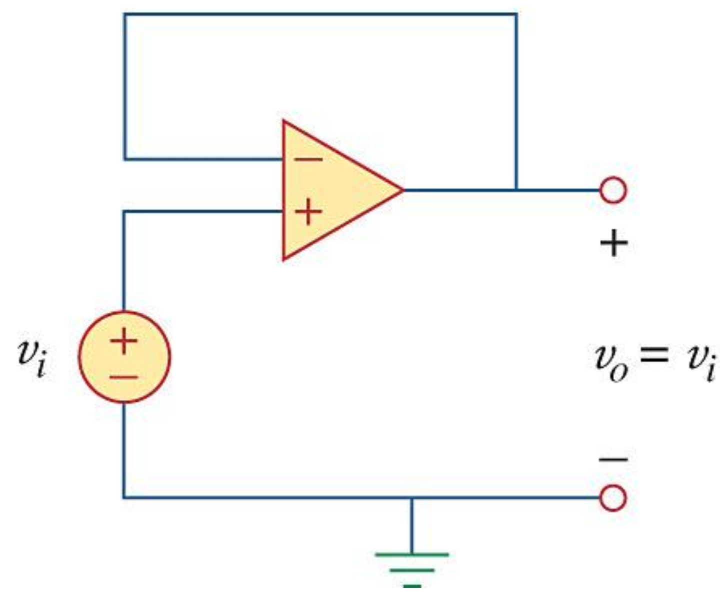
voltage follower

Ex

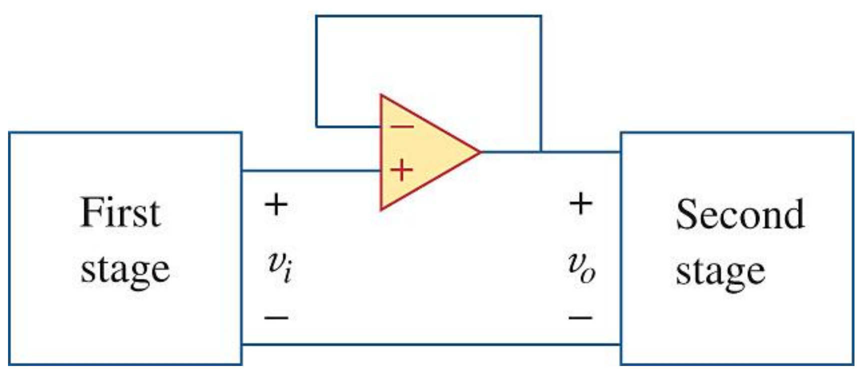


The voltage gain is $A_v = \frac{v_o}{v_i} = 1 + \frac{R_f}{R_1}$, which does not have a negative sign. Thus the output has the same polarity as the input.

Note: If $R_f = 0$ or $R_1 = \infty$, or both, the gain becomes 1. Under this conditions, the circuit becomes a *voltage follower* (The output follows the input).



A voltage follower is used to isolate two cascaded stages of a circuit.



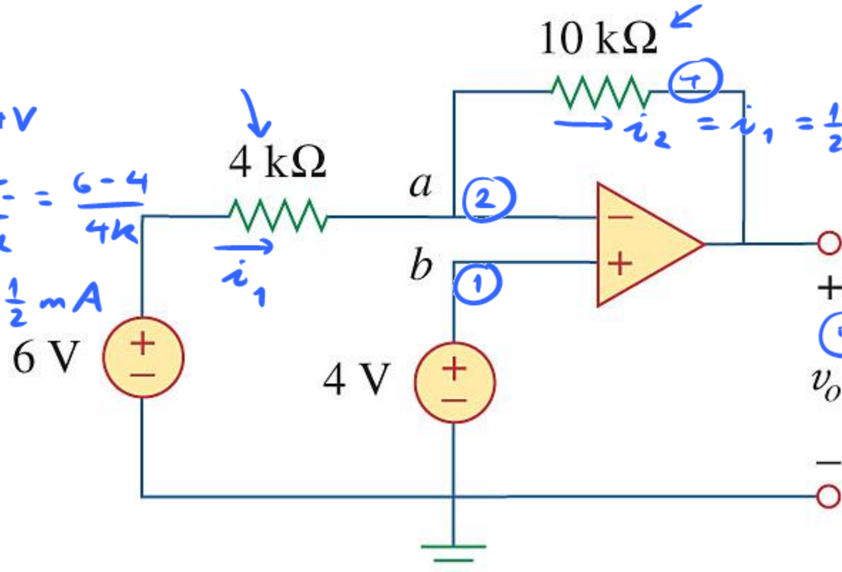
Ex. Calculate the output voltage v_o for the op amp circuit below.

Method 1

(1) $v_+ = 4V$

(2) $v_- = v_+ = 4V$

(3) $i_1 = \frac{6 - v_-}{4k} = \frac{6-4}{4k} = \frac{2}{4k} = \frac{1}{2} mA$



(4) $i_2 = i_1 = \frac{1}{2} mA$

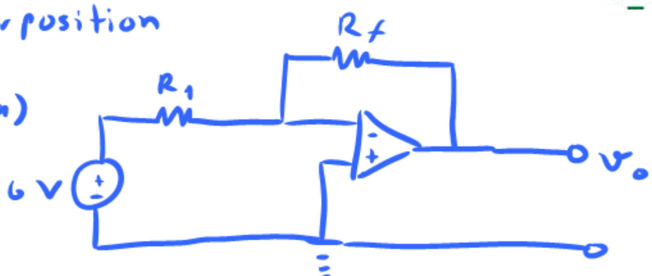
$i_2 = \frac{v_a - v_o}{10k}$

(5) $v_o = 4 - \frac{1}{2}m \times 10k = 4 - 5 = \boxed{-1\ V}$

Method 2

Superposition

a)



"6V" acting alone

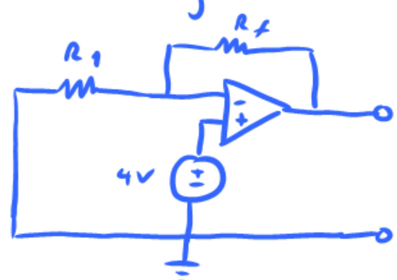
Inverting Amp.

$v_o = -\frac{R_f}{R_1} v_{in}$

$= -\frac{\cancel{10k}^{5}}{\cancel{4k}} \times \cancel{6}^{3} V$

$= -15V$

b) "4V" acting alone



"non-inverting amp."

$v_o = \left(1 + \frac{R_f}{R_1}\right) v_{in}$

$= \left(1 + \frac{10k}{4k}\right) \times 4$

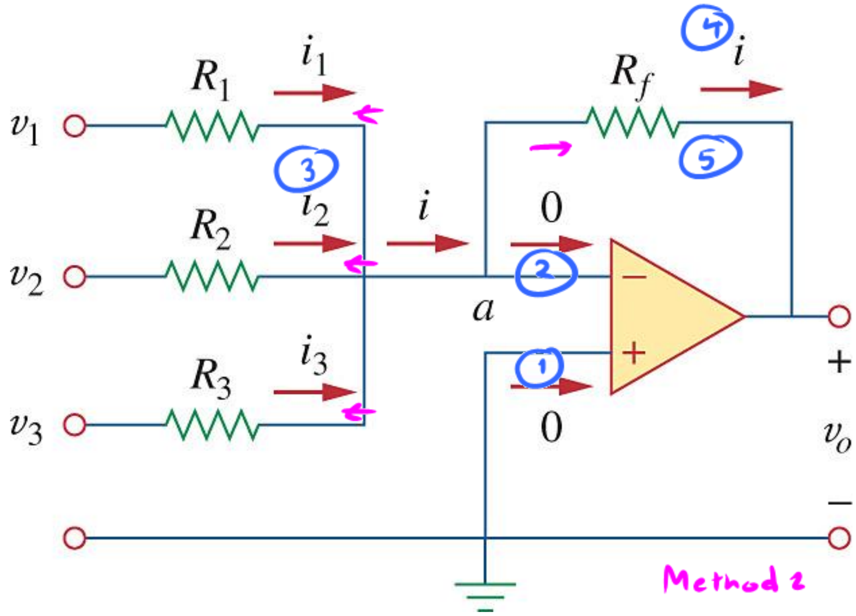
$= 4 + 10$

$= 14\ V$

$v_o = -15 + 14 = \boxed{-1\ V}$

5.4. Summing Amplifier

A summing amplifier is an op-amp circuit that combines several inputs and produces an output that is the weighted sum of the inputs. For this reason, the circuit is called a *summer*.



Method 2

KCL at "a"

$$\frac{\cancel{v_a}^{\,0} - v_1}{R_1} + \frac{\cancel{v_a}^{\,0} - v_2}{R_2} + \frac{\cancel{v_a}^{\,0} - v_3}{R_3} + \frac{\cancel{v_a}^{\,0} - v_o}{R_f} = 0$$

① $v_+ = 0\text{ V}$ because it is connected directly to ground.

② $v_- = v_+ = 0\text{ V}$

③ $i_1 = \dfrac{v_1 - v_-}{R_1} = \dfrac{v_1 - 0}{R_1} = \dfrac{v_1}{R_1}$, $i_2 = \dfrac{v_2}{R_2}$, $i_3 = \dfrac{v_3}{R_3}$ Ohm's law

④ $i = i_1 + i_2 + i_3$ = current through R_f

⑤ $v_o = v_a - iR_f = 0 - iR_f = -iR_f$

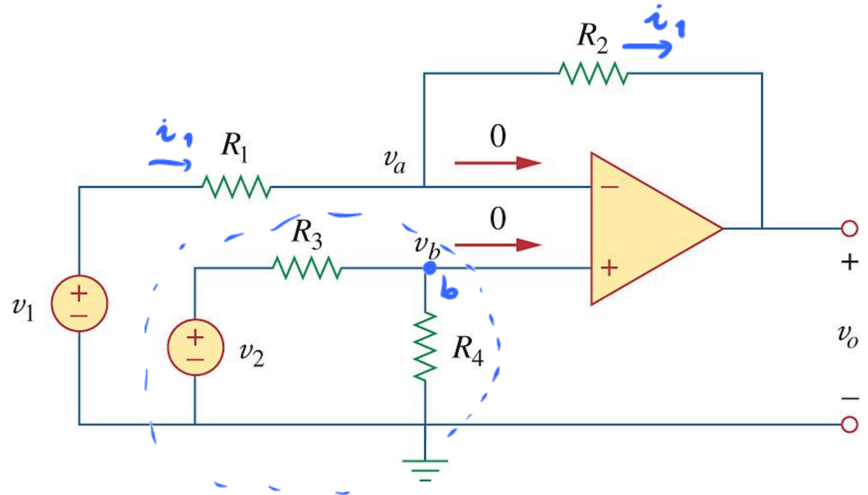
$$\underset{④}{=} -(i_1 + i_2 + i_3)R_f \underset{③}{=} -\left(\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3}\right)R_f$$

$$v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right).$$

Needless to say, the summer can have more that three inputs.

5.5. Difference Amplifier

A difference amplifier is a device that amplifies the difference between two inputs but rejects any signals common to the two inputs.



At node b,

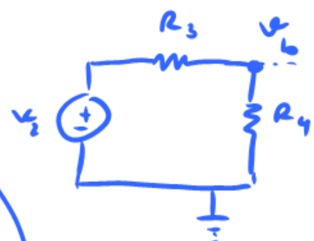
a) use voltage divider

$$v_b = \frac{R_4}{R_3+R_4} \times v_2$$

b) apply KCL at b

$$\frac{v_b - v_2}{R_3} + \frac{v_b}{R_4} = 0$$

$$v_b = \frac{1}{\frac{R_3}{R_4}+1} v_2 = \frac{1}{1+B} v_2, \quad B = \frac{R_3}{R_4}$$



$$v_a = v_- = v_+ = v_b = \frac{1}{1+B} v_2$$

a) $i_1 = \frac{v_1 - v_a}{R_1}$

A

$$v_o = v_a - i_1 \times R_2 = v_a - \frac{v_1 - v_a}{R_1} R_2$$

$$= v_a - (v_1 - v_a) \times A = v_a - A v_1 + A v_a$$

$$= -A v_1 + (1+A) v_a$$

$$= \boxed{-A v_1 + \frac{1+A}{1+B} v_2}$$

b) Apply KCL at "a"

$$v_o = \frac{R_2(1 + R_1/R_2)}{R_1(1 + R_3/R_4)} v_2 - \frac{R_2}{R_1} v_1$$

Since a difference amplifier must reject a signal common to the two inputs, the amplifier must have the property that $v_o = 0$ when $v_1 = v_2$. This property exists when

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}.$$

To actually make "difference" amp., I make $A = \frac{1+A}{1+B}$.

$\cancel{A} + AB = 1 + \cancel{A}$

$AB = 1$

In which case,

$v_o = -A v_1 + A v_2 = A(v_2 - v_1)$

Thus,

$$v_o = \frac{R_2}{R_1}(v_2 - v_1).$$

If $R_2 = R_1$ and $R_3 = R_4$, the difference amplifier becomes a *subtractor*, with the output

$$v_o = v_2 - v_1.$$

Ex. Design an op amp circuit with inputs v_1 and v_2 such that

$$v_o = -5v_1 + 3v_2.$$

Method 1

Use difference amp. $v_o = -A v_1 + \frac{1+A}{1+B} v_2$

Set $A = 5$

$\Rightarrow \frac{R_2}{R_1} = 5 \Rightarrow R_2 = 5 \times R_1$

Set $3 = \frac{1+5}{1+B} = \frac{6}{1+B}$

$1 + B = 2$

$B = 1$

$\Rightarrow \frac{R_3}{R_4} = 1 \Rightarrow R_3 = R_4$

$$\boxed{\text{Ans: Use difference amp. with } R_1 = 10\,k\Omega,\ R_2 = 50\,k\Omega,\ R_3 = R_4 = 20\,k\Omega}$$

New Q:

$v_o = -3v_1 + 5v_2$

$A = 3 = \frac{R_2}{R_1}$

$B = -\frac{1}{5} = \frac{R_3}{R_4}$

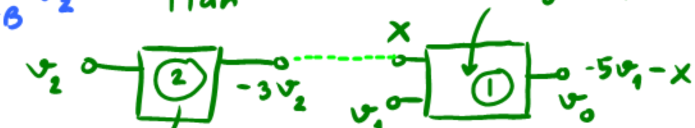
not possible to have.

So, can not use method 1.

However, can use method 2.

Method 2

Plan



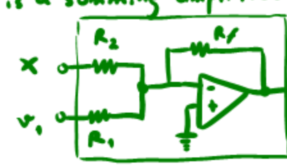
Observe that if we connect the output of box 1 into box 2, then we will have

$v_o = -5v_1 - X$

$= -5v_1 - (-3v_2)$

$= -5v_1 + 3v_2$ as required.

Box 1 is a summing amplifier:

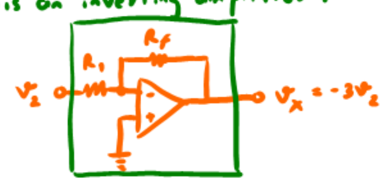


We need $\frac{R_f}{R_2} = 1$ and $\frac{R_f}{R_1} = 5$

so $R_f = R_2$ and $R_f = 5R_1$

Ex. Make $R_1 = 10\,k\Omega$, $R_2 = R_f = 50\,k\Omega$

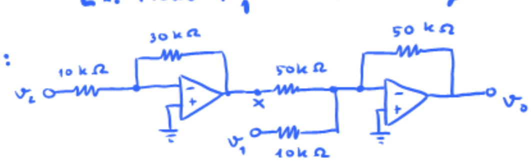
Box 2 is an inverting amplifier:



We need $\frac{R_f}{R_1} = 3$. Hence, $R_f = 3R_1$

Ex. Make $R_1 = 10\,k\Omega$ and $R_f = 30\,k\Omega$

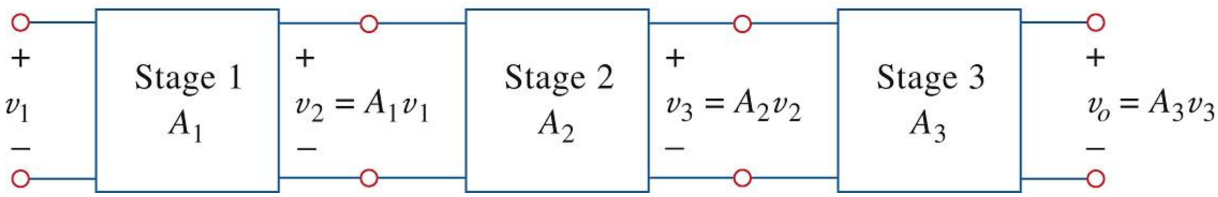
Ans:



5.6. Cascaded of Op Amp Circuits

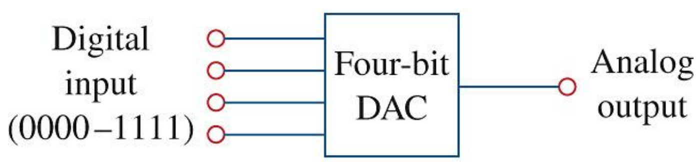
In practice, we can connect op amp circuits in cascade (i.e., head to tail) to achieve a large overall gain. Each circuit in the cascade is called **stage**. The output of one stage is the input to the next stage.

Op amp circuits have the advantage that they can be cascaded without changing their input-output relationships. This is due to the fact that each (ideal) op amp circuit has infinite input resistance and zero output resistance.

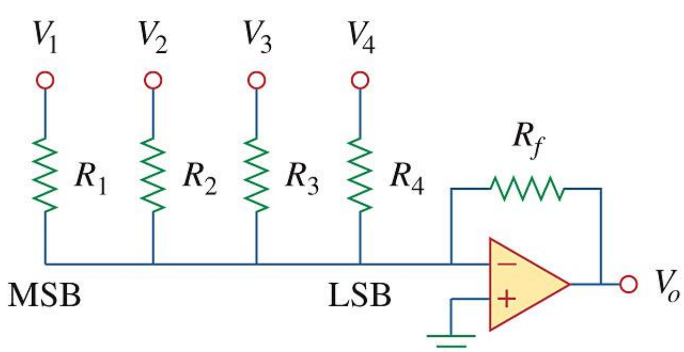


5.7. Application: Digital-to-Analog Converter (DAC)

The digital-to-analog converter (DAC) transforms digital signals into analog form. A typical example of a four-bit DAC is shown in (a) below.



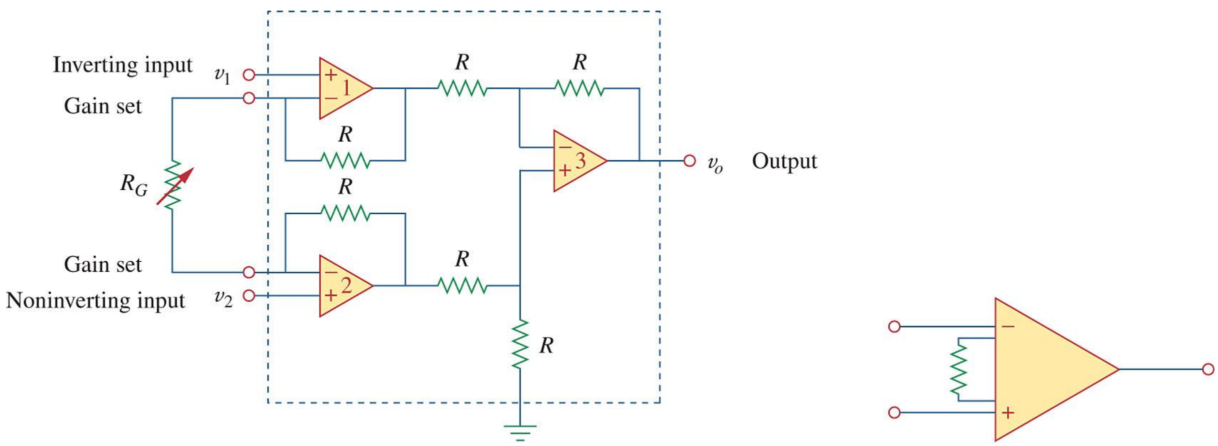
(a)



(b)

5.8. Application: Instrumentation(al) Amplifiers (IA)

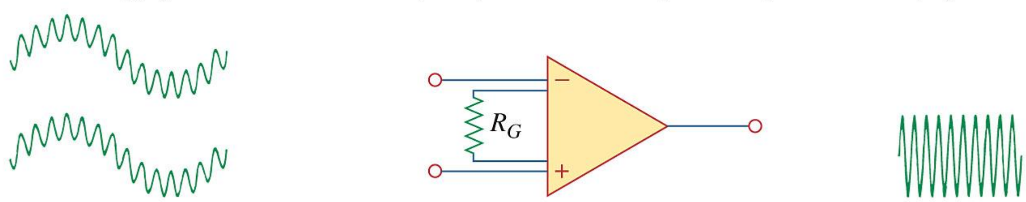
One of the most useful and versatile op amp circuits for precision measurement and process control. IA amplifies the difference between the input signals.



(a) (b)

$$v_0 = \left(1 + \frac{2R}{R_G}\right)(v_2 - v_1).$$

The instrumentation amplifier amplifies small differential signal voltages superimposed on larger common-mode voltages. Since the common-mode voltages are equal, they cancel each other.



Small differential signals riding on larger common-mode signals

Instrumentation amplifier

Amplified differential signal, No common-mode signal

The IA has three major characteristics:

(a) The voltage gain is adjusted by one external resistor R_G.

(b) The input impedance of both inputs is very high and does not vary as the gain is adjusted.

(c) The output v_o depends on the difference between the inputs, not on the voltage common to them.

Typical example of IA has gain from 1 to 1000.