Q1

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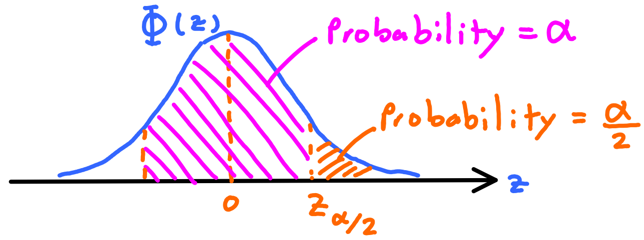
Recall that the confidence level of the interval estimate with limits

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

is given by $100(1-\alpha)\%$.

So, for parts (a)-(c), we first grab the values of $z_{\alpha/2}$ from the given interval.

2.14, 2.49, and 1.85



Note from the figure above that $\Phi(z_{\alpha/2}) = 1 - \frac{\alpha}{2}$.

Therefore, we can use the Φ table to find $1-\frac{\alpha}{2}$ and the value of α.

$1 - \frac{\alpha}{2} = p$

$\alpha = 2(1-p)$

	$z_{\alpha/2}$ $\rightarrow$	$1-\frac{\alpha}{2}$ $\rightarrow$	α $\rightarrow$	$100(1-\alpha)\%$
(a)	2.14	0.983823	0.0324	96.76%
(b)	2.49	0.993613	0.0128	98.72%
(c)	1.85	0.967843	0.0643	93.57%

For parts (d) - (f), we are given the confidence levels which are the same as $100(1-\alpha)\%$.

Therefore, we can calculate the values of $1-\frac{\alpha}{2}$ via

$$100(1-\alpha)\% = CL\%$$

$$1-\alpha = \frac{CL}{100}$$

$$\alpha = 1 - \frac{CL}{100}$$

$$\frac{\alpha}{2} = \frac{1}{2}\left(1 - \frac{CL}{100}\right)$$

$$1 - \frac{\alpha}{2} = 1 - \frac{1}{2} + \frac{CL}{100} = \frac{1}{2} + \frac{1}{2}\frac{CL}{100} \quad ★$$

Finally, we can use the Φ table to map the values of $1 - \frac{\alpha}{2}$ to the corresponding values of $z_{\alpha/2}$.

CL $\xrightarrow{★}$	$1 - \frac{\alpha}{2}$ $\xrightarrow{\Phi \text{ table}}$	$z_{\alpha/2}$
98%	0.99	2.33
80%	0.90	1.28
75%	0.875	1.15

Q2

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From confidence level (CL), we can get $1-\frac{\alpha}{2}$ from

$$\frac{1}{2}+\frac{1}{2}\frac{CL}{100}. \quad \star$$

For 95% CL, $1-\frac{\alpha}{2} = 0.975.$

90% CL, $1-\frac{\alpha}{2} = 0.995.$

From $1-\frac{\alpha}{2}$, we then use the Φ table to find $z_{\alpha/2}$.

the value of z at which $\Phi(z) = 1-\frac{\alpha}{2}$

$$CL \xrightarrow{\star} 1-\frac{\alpha}{2} \xrightarrow{\Phi \text{ table}} z_{\alpha/2}$$

CL	$1-\frac{\alpha}{2}$	$z_{\alpha/2}$
95%	0.975	1.96
90%	0.995	2.58

The confidence interval is given by

$[\ell, u]$ where $\ell = \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$u = \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

Here, $\sigma = 20$ and $\bar{x} = 1000$.

	CL	$z_{\alpha/2}$	n	ℓ	u	$u-\ell$
(a)	95%	1.96	10	987.6	1012.4	24.79
(b)	95%	1.96	25	992.2	1007.8	15.68
(c)	99%	2.58	10	983.7	1016.3	32.63
(d)	99%	2.58	25	989.7	1010.3	20.64

(e) $n\uparrow \Rightarrow$ length $\downarrow$ (narrower)

$CL\uparrow \Rightarrow$ length $\uparrow$ (wider)

(f) and (g)

(b)	95%	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
(c)	99%	2.58	10	983.7	1016.3	32.63
(d)	99%	2.58	25	989.7	1010.3	20.64

(e) $n \uparrow \Rightarrow$ length $\downarrow$ (narrower)

CL $\uparrow \Rightarrow$ length $\uparrow$ (wider)

(f) and (g)

The length of the CI is given by $2 z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

here, $w = 40$

If we want the length to be w, then we need

$$2 z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = w .$$

$$\Rightarrow \quad n = \left(2 z_{\alpha/2} \frac{\sigma}{w} \right)^2 \quad \star$$

	$z_{\alpha/2}$ $\xrightarrow{\star}$	n
(f)	1.96	$3.8416 \rightarrow 4$
(g)	2.58	$6.6564 \rightarrow 7$

min n such that width does not exceed 40.

note: Answer $n = 3$ if we want the maximum value of n such that the width does not fall below 40.

Q3

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(a)

From $n=11$ measured temperature, we can calculate $\bar{x}$ by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 13.77$$

It is given that $\sigma = 0.5$.

We want $CL = 99\%$. $\Rightarrow 1 - \frac{\alpha}{2} = \frac{1}{2}\left(1 + \frac{CL}{100}\right) = 0.995$

$$\Rightarrow z_{\alpha/2} = 2.58$$

The confidence interval is given by

$[l, u]$ where $l = \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 13.38$

$u = \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 14.16$

So, $13.38 \leq \mu \leq 14.16$.

(b)

The length of the CI is given by $2 z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

If we want the length to be w, then we need here, $w = 40$

$$2 z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = w.$$

$$\Rightarrow \qquad n = \left(2 z_{\alpha/2}\frac{\sigma}{w}\right)^2$$

At 95% CL, $z_{\alpha/2} = 1.96 \Longrightarrow n = 1.707 \rightarrow 2$

Q4-Q6

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Q4

$$CL = 100(1-\alpha) \Rightarrow \alpha = 1 - \frac{CL}{100} \Rightarrow \frac{\alpha}{2} = \frac{1}{2}\left(1 - \frac{CL}{100}\right) \star$$

CL $\xrightarrow{\star}$	$\alpha/2$	df $\xrightarrow{\text{t table}}$	$t_{\alpha/2,df}$
95%	0.0250	12	2.179
95%	0.0250	24	2.064
99%	0.0050	13	3.012
99.9%	0.0005	15	4.073

Q5

$$\left.\begin{array}{l} 95\%\ CL \xrightarrow{\star} \dfrac{\alpha}{2} = 0.025 \\ n = 16 \longrightarrow df = n-1 = 15 \end{array}\right\} \xrightarrow{\text{t table}} t_{\alpha/2,n-1} = 2.131$$

$\bar{x} = 60,139.7$ and $s = 3,645.94$

The confidence interval is given by $[l, u]$

where $$l = \bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = 58,197.3$$

$$u = \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = 62,082.1$$

Q6

From the sample, $$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \approx 231.67$$

$$s = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \approx 1.531$$

$$\left.\begin{array}{l} 90\%\ CL \xrightarrow{\star} \dfrac{\alpha}{2} = 0.05 \\ n = 5 \longrightarrow df = n-1 = 4 \end{array}\right\} \xrightarrow{\text{t table}} t_{\alpha/2,n-1} = 2.132$$

The confidence interval is given by $[\ell, u]$

where $\ell = \bar{x} - t_{\alpha/2, n-1} \dfrac{s}{\sqrt{n}} = 230.2$

$$u = \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} = 233.1$$

Q7

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For known σ, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

To find type I error, assume H_0 is true

So, $\mu = \mu_0 = 100$.

Hence, $\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

(a) $\alpha = P[\bar{X} \notin \text{acceptance region}] = 1 - P[98.5 \leq \bar{X} \leq 101.5]$

$$= 1 - P\left[\underbrace{\frac{98.5-100}{\sigma/\sqrt{n}}}_{-2.25} \leq \frac{\bar{X}-100}{\sigma/\sqrt{n}} \leq \underbrace{\frac{101.5-100}{\sigma/\sqrt{n}}}_{2.25}\right]$$

$$= 1 - (\Phi(2.25) - \Phi(-2.25))$$

$$= 1 - (\Phi(2.25) - (1 - \Phi(2.25)))$$

$$= 1 - (2\Phi(2.25) - 1) = 2(1 - \Phi(2.25))$$

$$= 0.02448$$

(b) When $n = 5$,

$$\alpha = 2(1 - \Phi(\underbrace{1.68}_{\frac{101.5-100}{\sigma/\sqrt{5}}})) = 0.09296$$

(c) Test statistic: $Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

acceptance region: $-z_{\alpha/2} \leq \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \leq z_{\alpha/2}$

$$\underbrace{\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{l} \leq \bar{X} \leq \underbrace{\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}_{u}$$

	α	$\rightarrow 1-\alpha/2$	$\xrightarrow{\Phi \text{ table}} z_{\alpha/2}$	n	$\rightarrow l$	u
(i)	0.01	0.995	2.58	9	98.28	101.72
(ii)	0.05	0.975	1.96	9	98.69	101.31
(iii)	0.01	0.995	2.5	5	97.69	102.31
(iv)	0.05	0.975	1.96	5	98.25	101.75

(d) To find β, the probability of type II error,

we assume that $\mu = 103$.

Again $\bar{X} \sim N(\mu, \sigma^2/n)$

$$\beta = P[\bar{X} \in \text{acceptance region}]$$
$$= P[\ l \le \bar{X} \le u\ ]$$
$$= P\left[\frac{l-\mu}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le \frac{u-\mu}{\sigma/\sqrt{n}}\right]$$
$$= \Phi\left(\frac{u-\mu}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{l-\mu}{\sigma/\sqrt{n}}\right)$$

	α	n	l	u	$\frac{l-\mu}{\sigma/\sqrt{n}}$	$\frac{u-\mu}{\sigma/\sqrt{n}}$	β
(i)	0.01	9	98.28	101.72	-7.08	-1.92	0.027429
(ii)	0.05	9	98.69	101.31	-6.46	-2.54	0.005543
(iii)	0.01	5	97.69	102.31	-5.93	-0.77	0.220650
(iv)	0.05	5	98.25	101.75	-5.31	-1.39	0.082264

(e) Same n : $\alpha \uparrow \Rightarrow \beta \downarrow$

Same α : $n \uparrow \Rightarrow \beta \downarrow$

(f) Recall that $p = 2(1 - \Phi(|z_0|))$

$z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

	$\bar{x}$	z_0	p
(i)	98	-3	0.002700
(ii)	101	1.5	0.133614
(iii)	102	3	0.002700

Q8

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(a)

$\alpha = 0.05 \Rightarrow z_{\alpha/2} = 1.96$

(We have seen the $z_{\alpha/2}$ for $\alpha=0.05$ many times.)

$$z_0 = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{2.78-3}{0.9/\sqrt{15}} = -0.9467$$

Because $-0.9467 \in [-z_{\alpha/2}, z_{\alpha/2}]$, we fail to reject H_0.

(There is not enough evidence to support the claim that the mean differs from 3 at $\alpha = 0.05$)

(b) $P = 2(1-\Phi(|z_0|)) = 0.3438$

(Again, because $\alpha < P$, we fail to reject H_0.)

(c) If the true mean is $\mu = 3.25$, then

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

The test statistic is $Z_0 = \dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

Power of the test $= 1-\beta =$ the probability of correctly $\underbrace{\text{rejecting}}_{Z_0 \text{ is outside } [-z_{\alpha/2}, z_{\alpha/2}]}$ H_0

$$= 1 - P\left[Z_0 \in [-z_{\alpha/2}, z_{\alpha/2}]\right]$$

$$= 1 - P\left[-z_{\alpha/2} \le \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right]$$

$$= 1 - P\left[\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{X} \le \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

$$= 1 - P\left[\underbrace{\frac{\mu_0-\mu}{\sigma/\sqrt{n}} - z_{\alpha/2}}_{-3.0358} \le \underbrace{\boxed{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}}_{\mathcal{N}(0,1)} \le \underbrace{\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + z_{\alpha/2}}_{0.8842}\right]$$

$$= 1 - \underbrace{\left(\Phi(0.88) - \Phi(-3.04)\right)}_{\beta = 0.8094} = 0.1906$$

Q9-Q11

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Q9

Critical values $= \pm t_{\alpha/2, n-1}$

α	$\alpha/2$	$n-1$	$\pm t_{\alpha/2,n-1}$
0.01	0.005	19	± 2.861
0.05	0.025	11	± 2.201
0.10	0.05	14	± 1.761

Q10

This is the value of α in the t table where $t_{\alpha/2,n-1} = t_0$

$$p = 2\left(\underbrace{1 - F_{T_{n-1}}(|t_0|)}\right)$$

instead of using Φ, which is the cdf of $\mathcal{N}(0,1)$, here, we need to use the cdf of T_{n-1}

Here, $df = n-1$. So, we look at the row with $df = n-1$ in the t table. Usually, we will not be able to find the value of t that is exactly the same as $|t_0|$ in the row. So, we will take the nearest t's (one $< |t_0|$ and another $> |t_0|$). These two t values then gives two values of α which bound $1 - F_{T_{n-1}}(|t_0|)$

$\hookrightarrow \alpha_{high}$ and α_{low}

We then conclude that $2\alpha_{low} < p < 2\alpha_{high}$

	t_0	$\|t_0\|$	t_{low}	t_{high}	α_{high}	α_{low}	
(a)	2.05	2.05	1.729	2.093	0.05	0.025	$\Rightarrow 0.05 < p < 0.1$
(b)	-1.84	1.8	1.729	2.093	0.05	0.025	$\Rightarrow 0.05 < p < 0.1$
(c)	0.4	0.4	0.257	0.688	0.40	0.25	$\Rightarrow 0.5 < p < 0.8$

Remark : With MATLAB, the values of p can be calculated. They are 0.0544, 0.0814, 0.6936, respectively.

Q11

$H_0: \mu = 300 \leftarrow \mu_0$

$H_1: \mu \neq 300$

From the sample, we calculate $\bar{x} = 325.4963$

From the sample, we calculate $\bar{x} = 325.4963$

$s = 198.7855$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = 0.6665$$

With $\alpha = 0.05$, the critical values are $\pm t_{\alpha/2, n-1} = \pm 2.056$

Because $t_0 \in [-2.056, +2.056]$, we fail to reject H_0.

Q12

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Recall the simple linear regression model:

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

(a)

The least squares estimates of the slope ($\hat{\beta}_1$) and the intercept ($\hat{\beta}_0$) are given by

$$\hat{\beta}_1 = \frac{\overline{xy} - (\bar{x}\,\bar{y})}{\overline{x^2} - (\bar{x})^2} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} \approx -2.3298$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i\right) \approx 48.013$$

(b) $\hat{\sigma}^2 = \dfrac{SS_E}{n-2}$

$$\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = -59.0571$$

$$SS_E = S_{yy} - \hat{\beta}_1 S_{xy} \approx 22.1299$$

$$\sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n} = 159.7143$$

$$\hat{\sigma}^2 = \frac{SS_E}{n-2} \approx 1.8436$$

Fitted line: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

(c) With $x = 4.3$, $\hat{y} \approx 37.9948$

(d) with $x = 3.7$, $\hat{y} \approx 39.3927$

(e) Residual $= e = y - \hat{y} = 46.1 - 39.3927 = 6.7073$