HW Solution 5 — March 16

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) You may use programs such as `MATLAB` or Microsoft Excel to obtain your answers. You may also want to think about how to use your calculator to calculate the quantities asked in this HW in the case that similar questions show up on the final exam.

Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$
\begin{array}{c|ccc}
 & y \quad 2 & 4 & 5 \\
x & & & \\
\hline
1 & 0.02 & 0.10 & 0.08 \\
3 & 0.08 & 0.32 & 0.40
\end{array}
$$

(a) Find the following quantities:

    (i) the marginal pmf $p_X(x)$

    (ii) the marginal pmf $p_Y(y)$

    (iii) $\mathbb{E}X$

    (iv) $\operatorname{Var} X$

    (v) $\mathbb{E}Y$

    (vi) $\operatorname{Var} Y$

    (vii) $\mathbb{E}[XY]$

   (viii) $\mathbb{E}\left[(X-3)(Y-2)\right]$

   (ix) $\mathbb{E}\left[X(Y^3 - 11Y^2 + 38Y)\right]$

   (x) $\mathrm{Cov}\left[X, Y\right]$

   (xi) $\rho_{X,Y}$

   (xii) $\rho_{X,X}$

(b) One important property of covariance is that

$$\mathrm{Cov}\left[aX + b, cY + d\right] = ac\,\mathrm{Cov}\left[X, Y\right]. \tag{5.1}$$

To see this, we write

$$\begin{aligned}
\mathrm{Cov}\left[aX + b, cY + d\right] &= \mathbb{E}\left[((aX+b) - \mathbb{E}\left[aX+b\right])((cY+d) - \mathbb{E}\left[cY+d\right])\right] \\
&= \mathbb{E}\left[((aX+b) - (a\mathbb{E}X+b))((cY+d) - (c\mathbb{E}Y+d))\right] \\
&= \mathbb{E}\left[(aX - a\mathbb{E}X)(cY - c\mathbb{E}Y)\right] \\
&= ac\mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right] \\
&= ac\,\mathrm{Cov}\left[X, Y\right].
\end{aligned}$$

With the help of (5.1), calculate the following quantities using the values of $\mathrm{Var}\,X$, $\mathrm{Cov}\left[X, Y\right]$, and $\rho_{X,Y}$ that you got from part (a).

   (i) $\mathrm{Cov}\left[3X + 4, 6Y - 7\right]$

   (ii) $\rho_{3X+4,6Y-7}$

   (iii) $\mathrm{Cov}\left[X, 6X - 7\right]$

   (iv) $\rho_{X,6X-7}$

Solution:

(a)

   (i) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(iii) From `Excel`, $\mathbb{E}X = 2.6$

(iv) From `Excel`, $\operatorname{Var} X = 0.64$

(v) From `Excel`, $\mathbb{E}Y = 4.28$

(vi) From `Excel`, $\operatorname{Var} Y = 0.8016$

(vii) From `Excel`, $\mathbb{E}[XY] = 11.16$

(viii) From `Excel`, $\mathbb{E}[(X-3)(Y-2)] = -0.88$

(ix) From `Excel`, $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = 104$

(x) From `Excel`, $\operatorname{Cov}[X, Y] = 0.032$

(xi) From `Excel`, $\rho_{X,Y} = 0.044677$

(xii) $\rho_{X,X} = \frac{\operatorname{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\operatorname{Var}[X]}{\sigma_X^2} = 1$

(b)

(i) $\operatorname{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \operatorname{Cov}[X, Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}$.

(ii) Note that

$$\rho_{aX+b,cY+d} = \frac{\operatorname{Cov}[aX + b, cY + d]}{\sigma_{aX+b}\sigma_{cY+d}}$$
$$= \frac{ac\operatorname{Cov}[X, Y]}{|a|\sigma_X |c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \operatorname{sign}(ac) \times \rho_{X,Y}.$$

Hence, $\rho_{3X+4,6Y-7} = \operatorname{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

(iii) $\operatorname{Cov}[X, 6X - 7] = 1 \times 6 \times \operatorname{Cov}[X, X] = 6 \times \operatorname{Var}[X] \approx \boxed{3.84}$.

(iv) $\rho_{X,6X-7} = \operatorname{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(i)$ and $p_Y(j)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\mathrm{Var}\,X$

(e) Find $\mathrm{Var}\,Y$

(f) Find $\mathbb{E}\,[XY]$

(g) Find $\mathbb{E}\,[(3X+4)(6Y-7)]$

(h) Find $\mathrm{Cov}\,[X,Y]$

(i) Find $\mathrm{Cov}\,[3X+4, 6Y-7]$

    Solution: All of the calculations in this question are simply plugging in numbers into appropriate formula.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

    The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) From `Excel`, $\mathbb{E}X = 1.9155$

(c) From `Excel`, $\mathbb{E}Y = 3.1268$

(d) From `Excel`, $\mathrm{Var}\,X = 0.6407$

(e) From `Excel`, $\mathrm{Var}\,Y = 1.8290$

(f) From `Excel`, $\mathbb{E}[XY] = 6.0282$

(g) From `Excel`, $\mathbb{E}[(3X + 4)(6Y - 7)] = 115.3239$

(h) From `Excel`, $\text{Cov}[X, Y] = 0.0389$

(i) $\text{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \text{Cov}[X, Y] = 0.6999$.

Problem 3. Least Square Error

(a) Suppose you have 8 observations:

$$-10, -9, 1, 2, 3, 3, 4, 5.$$

Let's denote these values by $x_1, x_2, \ldots, x_8$. You want to report one number which best represents this collection of observations. Let's call this number b.

  (i) Consider the total square error:

$$e_1(b) = \sum_{k=1}^{8} (x_i - b)^2.$$

     i. Plot $e_1(b)$ as a function of b from $b = -10$ to $b = 5$.
    ii. Find the value of b that minimizes $e_1(b)$.

  (ii) Consider the total absolute error:

$$e_2(b) = \sum_{k=1}^{8} |x_i - b|.$$

     i. (Difficult) Plot $e_2(b)$ as a function of b from $b = -10$ to $b = 5$.
    ii. (Difficult) Find the values of b that minimize $e_2(b)$.

(b) Suppose you have 10 pairs of observations:

$$(5, 11), (1, 4), (3, 7), (10, 21), (2, 5), (9, 19), (6, 14), (10, 22), (1, 4), (5, 11).$$

Let's denote these pairs of values by $(x_1, y_1), (x_2, y_2), \ldots, (x_{10}, y_{10})$. You want to report the relationship between the x and the y by a linear (affine) expression $y = ax + b$. Find the values of a and b that minimize

$$\sum_{k=1}^{10} (y_k - ax_k - b)^2.$$

Solution:

(a)

(i)

　　i. See Figure 5.1.

　　ii. $b = \bar{x} = -0.1250$ minimizes $e_1(b)$.

(ii)

　　i. See Figure 5.1.

　　ii. Any b in the interval $[2, 3]$ minimizes $e_2(b)$. (Note that the median is 2.5.)
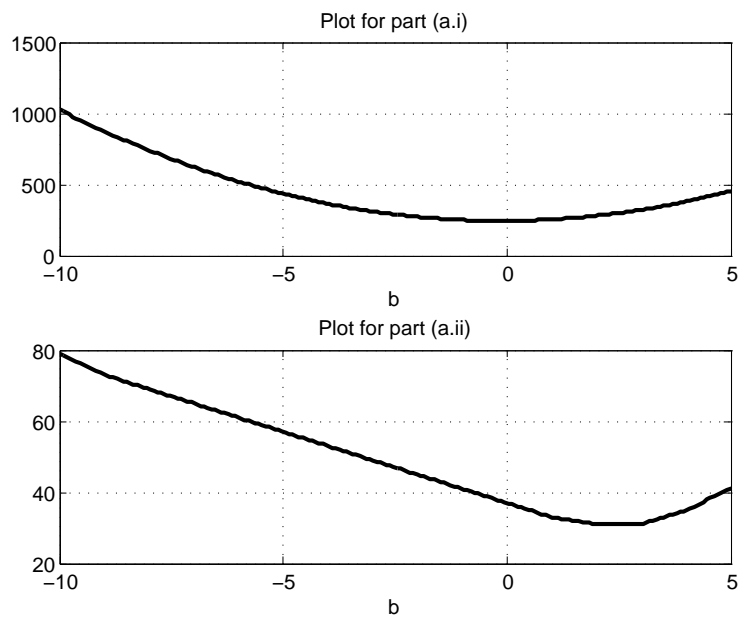
(b) From `Excel`, $a = 1.9749$ and $b = 1.5305$.

Figure 5.1: Plots for problem 3.