Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c|c} x \backslash y & \begin{array}{ccc} 2 & 4 & 6 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 7 \end{array} & \left[\begin{array}{ccc} 0.1 & 0.1 & 0 \\ 0.2 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{array}\right] \end{array}$$

$0.1 + 0.1 + 0 = 0.2$
$0.2 + 0.3 + 0 = 0.5$
$0 + 0 + 0.3 = 0.3$
1

$p_{X,Y}(4,2) = P[X = 4 \text{ and } Y = 2]$

Find the following quantities:

(a) the marginal pmf $p_X(x)$

$$p_X(x) = \begin{cases} 0.2, & x = 3, \\ 0.5, & x = 4, \\ 0.3, & x = 7, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbb{E}X = \sum_x x\, p_X(x) = \overset{0.6}{0.2 \times 3} + \overset{2.0}{0.5 \times 4} + \overset{2.1}{0.3 \times 7} = 4.7$

(b) the marginal pmf $p_Y(y)$

$0.1 + 0.2 + 0 = 0.3$
$0.1 + 0.3 + 0 = 0.4$
$0 + 0 + 0.3 = 0.3$
1

$$p_Y(y) = \begin{cases} 0.3, & y = 2, \\ 0.4, & y = 4, \\ 0.3, & y = 6, \\ 0, & \text{otherwise.} \end{cases}$$

(c) $\mathbb{E}X = 4.7$

(d) $\operatorname{Var} X = \underbrace{\mathbb{E}[X^2]}_{24.5} - (\mathbb{E}X)^2 = 2.41$

(e) $\mathbb{E}Y = 4$

(f) $\operatorname{Var} Y = 2.4$

(g) $\mathbb{E}[XY] = 20.8$

$\mathbb{E}[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y)$

(h) $\mathbb{E}[(X - Y^3)(X + 1)] = -622$

(i) $\operatorname{Cov}[X, Y]$

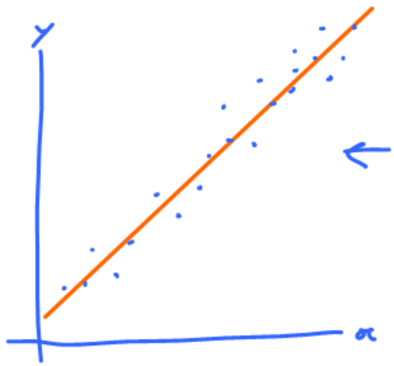
$= \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$

$= 20.8 - \underbrace{(4.7)(4)}_{18.8}$

$= 2$

Problem 2. Suppose you make 13 pairs of observations:

x	y
1	14.1338
4	32.3236
1	5.9754
5	35.785
7	52.4688
6	49.5751
10	77.6489
2	13.5761
6	49.7059
3	28.5717
7	51.8538
3	27.0028
4	27.4189



$x \longrightarrow ax+b$

vs.

y

error $= y - (ax+b)$

Let's denote these pairs of values by $(x_1, y_1), (x_2, y_2), \ldots, (x_{13}, y_{13})$. You want to report the relationship between the x and the y by a linear (affine) expression $y = ax + b$. Find the values of a and b that minimize

$$\sum_{k=1}^{13} (\underbrace{y_k - ax_k - b}_{\text{error}})^2.$$

$$a^* = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - (\bar{x})^2}$$

$$b^* = \bar{y} - a^*\bar{x}$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\overline{x^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\overline{xy} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

Excel

$a^* \approx 7.44$

$b^* \approx 2.08$