**IES 302: Engineering Statistics** **2011/2**

# HW Solution 2 — Due: February 8

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### Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1.** The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let $A$ denote the event $\{a, b, c\}$, and let $B$ denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

***Solution***:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned}P(B) = P(\{c,d,e\}) &= P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.}\end{aligned}$$

(c) $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

**Problem 2.**

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of the possible value for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of the possible value for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

***Solution***:

(a) We will first try to bound $P(A \cap B)$. Note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. To summarize, we now know that

$$P(A \cap B) \leq \min\{P(A), P(B)\}.$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Applying the fact that $P(A \cup B) \leq 1$, we then have

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

If the number of the RHS is $> 0$, then it is a new information. However, if the number on the RHS is negative, it is useless and we will use the fact that $P(A \cap B) \geq 0$. To summarize, we now know that

$$\max\{P(A) + P(B) - 1, 0\} \leq P(A \cap B).$$

In conclusion,

$$\max\{(P(A)+P(B)-1),0\} \leq P(A\cap B) \leq \min\{P(A),P(B)\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6},\frac{1}{2}\right]}$. The upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) By monotonicity we must have

$$P(A\cup B) \geq \max\{P(A),P(B)\}.$$

On the other hand, we know that

$$P(A\cup B) \leq P(A)+P(B).$$

If the RHS is $> 1$, then the inequality is useless and we simply use the fact that it must be $\leq 1$. To summarize, we have

$$P(A\cup B) \leq \min\{(P(A)+P(B)),1\}.$$

In conclusion,

$$\max\{P(A),P(B)\} \leq P(A\cup B) \leq \min\{(P(A)+P(B)),1\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$, we have

$$P(A\cup B) \in \boxed{\left[\frac{1}{2},\frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.

**Problem 3.** Let $A$ and $B$ be events for which $P(A)$, $P(B)$, and $P(A\cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A\cap B)$

(b) $P(A\cap B^c)$

(c) $P(B\cup(A\cap B^c))$

(d) $P(A^c \cap B^c)$

***Solution***:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

**Problem 4.**

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

***Solution***:

(a) Recall that $P(A \cap B) = P(A|B)P(B)$. Therefore,

(i) $P(A \cap B) = 0.4 \times 0.5 = \boxed{0.2.}$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, $P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3$.

(b) By the total probability formula, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$

**Problem 5.** [Gubner, 2006, Q2.60] You have five computer chips, two of which are known to be defective.

(a) You test one of the chips; what is the probability that it is defective?

(b) Your friend tests two chips at random and reports that one is defective and one is not. Given this information, you test one of the three remaining chips at random; what is the conditional probability that the chip you test is defective?

***Solution***:

(a) $\boxed{\frac{2}{5}}$ (two of five chips are defective.)

(b) Among the three remaining chips, only one is defective. So, the conditional probability that the chosen chip is defective is $\boxed{\frac{1}{3}.}$

**Problem 6.** Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped.

(a) Find the probability that a packet is dropped.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.

[Gubner, 2006, Ex.1.20]

***Solution***: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1 - 3/4) \\ &= 1/4 + 1/16 = \boxed{5/16}. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E\cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}}$.

**Problem 7.** You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

***Solution***: Let $F, U$, and $H$ be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}} = \frac{1}{1+\frac{2}{3}} = \boxed{\frac{3}{5}}.$$

**Problem 8.** You have three coins in your pocket, two fair ones but the third biased with probability of heads $p$ and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

***Solution***: Let $F, U$, and $H$ be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3}+p\times\frac{1}{3}} = \boxed{\frac{1}{1+p}}.$$

**Problem 9.** Someone has rolled a fair die twice. You know that one of the rolls turned up a face value of six. What is the probability that the other roll turned up a six as well?

Hint: Not $\frac{1}{6}$.

***Solution***: Take as sample space the set $\{(i,j)|i,j = 1,\ldots,6\}$, where $i$ and $j$ denote the outcomes of the first and second rolls. A probability of 1/36 is assigned to each element of the sample space. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6), \ldots, (5,6), (6,6), (6,5), \ldots, (6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$. [Tijms, 2007, Example 8.1, p. 244]

**Problem 10.** An article in the British Medical Journal ["Comparison of Treatment of Renal Calculi by Operative Surgery, Percutaneous Nephrolithotomy, and Extracorporeal Shock Wave Lithotripsy" (1986, Vol. 82, pp. 879892)] provided the following discussion of success rates in kidney stone removals. Open surgery (OS) had a success rate of 78% (273/350) while a newer method, percutaneous nephrolithotomy (PN), had a success rate of 83% (289/350). This newer method looked better, but the results changed when stone diameter was considered. For stones with diameters less than two centimeters, 93% (81/87) of cases of open surgery were successful compared with only 87% (234/270) of cases of PN. For stones greater than or equal to two centimeters, the success rates were 73% (192/263) and 69% (55/80) for open surgery and PN, respectively. Open surgery is better for both stone sizes, but less successful in total. In 1951, E. H. Simpson pointed out this apparent contradiction (known as Simpsons Paradox) but the hazard still persists today. Explain how open surgery can be better for both stone sizes but worse in total. [Montgomery and Runger, 2010, Q2-115]

***Solution***: First, let's recall the total probability theorem:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)\,P(B) + P(A|B^c)\,P(B^c). \end{aligned}$$

We can see that $P(A)$ does not depend only on $P(A \cap B)$ and $P(A|B^c)$. It also depends on $P(B)$ and $P(B^c)$. In the extreme case, we may imagine the case with $P(B) = 1$ in which $P(A) = P(A|B)$. At another extreme, we may imagine the case with $P(B) = 0$ in which $P(A) = P(A|B^c)$. Therefore, depending on the value of $P(B)$, the value of $P(A)$ can be anywhere between $P(A|B)$ and $P(A|B^c)$.

Now, let's consider events $A_1$, $B_1$, $A_2$, and $B_2$. Let $P(A_1|B_1) = 0.93$ and $P(A_1|B_1^c) = 0.73$. Therefore, $P(A_1) \in [0.73, 0.93]$. On the other hand, let $P(A_2|B_2) = 0.87$ and $P(A_2|B_2^c) = 0.69$. Therefore, $P(A_2) \in [0.69, 0.87]$. With small value of $P(B_1)$, the value of $P(A_1)$ can be 0.78 which is closer to its lower end of the bound. With large value of $P(B_2)$, the value of $P(A_2)$ can be 0.83 which is closer to its upper end of the bound. Therefore, even though $P(A_1|B_1) > P(A_2|B_2) = 0.87$ and $P(A_1|B_1^c) > P(A_2|B_2^c)$, it is possible that $P(A_1) < P(A_2)$.

In the context of the paradox under consideration, note that the success rate of PN with small stones (87%) is higher than the success rate of OS with large stones (73%). Therefore, by having a lot of large stone cases to be tested under OS and also have a lot of small stone cases to be tested under PN, we can create a situation where the overall success rate of PN comes out to be better then the success rate of OS. This is exactly what happened in the study as shown in Table 2.1.

Open surgery

| | success | failure | sample size | sample percentage | conditional success rate |
|---|---|---|---|---|---|
| large stone | 192 | 71 | 263 | 75% | 73% |
| small stone | 81 | 6 | 87 | 25% | 93% |
| overall summary | 273 | 77 | 350 | 100% | 78% |

PN

| | success | failure | sample size | sample percentage | conditional success rate |
|---|---|---|---|---|---|
| large stone | 55 | 25 | 80 | 23% | 69% |
| small stone | 234 | 36 | 270 | 77% | 87% |
| overall summary | 289 | 61 | 350 | 100% | 83% |

Table 2.1: Success rates in kidney stone removals.