**IES 302: Engineering Statistics** **2011/2**

# HW Solution 1 — Due: February 1

*Lecturer: Prapun Suksompong, Ph.D.*

## Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

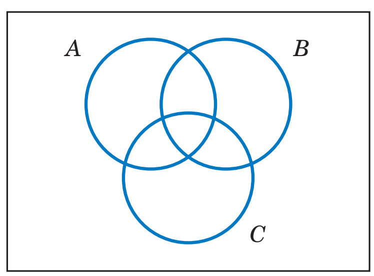
(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1.** (**Set Theory**)

(a) Three events are shown on the Venn diagram in the following figure:




Reproduce the figure and shade the region that corresponds to each of the following events.

(i) $A^c$

(ii) $A \cap B$

(iii) $(A \cap B) \cup C$

(iv) $(B \cup C)^c$

(v) $(A \cap B)^c \cup C$

[Montgomery and Runger, 2010, Q2-19]

(b) Let $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and put $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6\}$. Find $A \cup B$, $A \cap B$, $A \cap C$, $A^c$, and $B \setminus A$.

For this problem, only answers are needed; you don't have to describe your solution.

***Solution***:

(a) See Figure 1.1

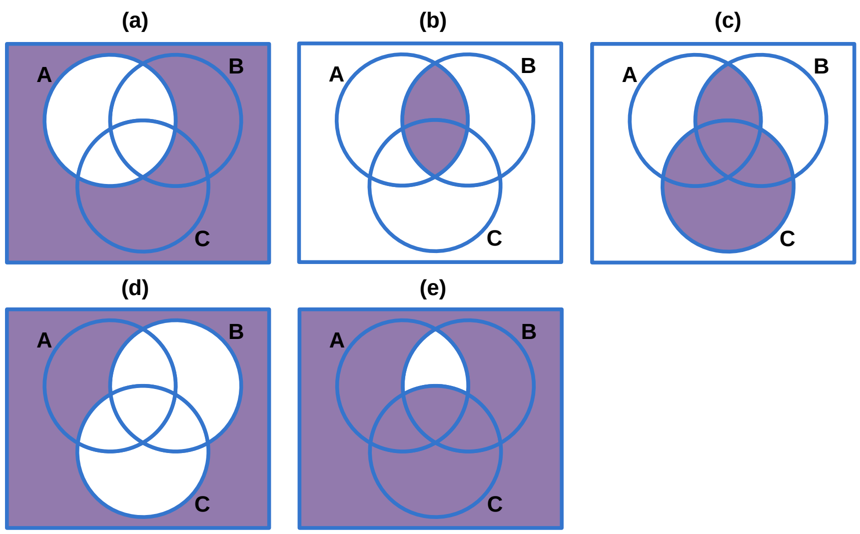



Figure 1.1: Venn diagrams for events in Problem 1

(b) $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3, 4\}$, $A \cap C = \emptyset$, $B \setminus A = \{5, 6\} = C$.

**Problem 2.** (**Classical Probability**) There are three buttons which are painted red on one side and white on the other. If we tosses the buttons into the air, calculate the probability that all three come up the same color.

Remarks: A *wrong* way of thinking about this problem is to say that there are four ways they can fall. All red showing, all white showing, two reds and a white or two whites and a red. Hence, it seems that out of four possibilities, there are two favorable cases and hence the probability is 1/2.

***Solution***: There are 8 possible outcomes. (The same number of outcomes as tossing three coins.) Among these, only two outcomes will have all three buttons come up the same color. So, the probability is $2/8 = \boxed{1/4}$.

**Problem 3.** (**Classical Probability**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

***Solution***:

(a) By the multiplication rule, total number of possible designs

$$= 4 \times 3 \times 5 \times 3 \times 5 = \boxed{900}.$$

(b) From part (a), total number of possible designs is 900. The sample space is now the set of all possible designs that may be seen on five visits. It contains $(900)^5$ outcomes. (This is ordered sampling with replacement.)

The number of outcomes in which all five visits are different can be obtained by realizing that this is ordered sampling without replacement and hence there are $(900)_5$ outcomes. (Alternatively, On the first visit any one of 900 designs may be seen. On the second visit there are 899 remaining designs. On the third visit there are 898 remaining designs. On the fourth and fifth visits there are 897 and 896 remaining designs, respectively. From the multiplication rule, the number of outcomes where all designs are different is $900 \times 899 \times 898 \times 897 \times 896$.)

Therefore, the probability that a design is not seen again is

$$\frac{(900)_5}{900^5} \approx \boxed{0.9889.}$$

**Problem 4.** (**Combinatorics**) Consider the design of a communication system in the United States.

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?

(b) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

(c) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?

[Montgomery and Runger, 2010, Q2-45]

***Solution***:

(a) From the multiplication rule (or by realizing that this is ordered sampling with replacement), $10^3 = \boxed{1,000}$ prefixes are possible

(b) This is ordered sampling without replacement. Therefore $(10)_3 = 10 \times 9 \times 8 = \boxed{720}$ prefixes are possible

(c) From the multiplication rule, $8 \times 2 \times 10 = \boxed{160}$ prefixes are possible.

**Problem 5.** (**Classical Probability**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

***Solution***: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

**Problem 6.** (**Classical Probability**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

***Solution***: There are $2^{10}$ possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails. (Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

**Problem 7.** (**Classical Probability**) Shuffle a deck of cards and cut it into three piles. What is the probability that (at least) a court card will turn up on top of one of the piles.

Hint: There are 12 court cards (four jacks, four queens and four kings) in the deck.

***Solution***: In [Lovell, 2006, p. 17–19], this problem is named "Three Lucky Piles". When somebody cuts three piles, they are, in effect, randomly picking three cards from the deck. There are $52 \times 51 \times 50$ possible outcomes. The number of outcomes that do not contain any court card is $40 \times 39 \times 38$. So, the probability of having at least one court card is

$$\boxed{\frac{52 \times 51 \times 50 - 40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.}$$

**Problem 8.** ***Binomial theorem***: For any positive integer $n$, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{1.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (1.1) to evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$.

***Solution***:

(a) $\binom{25}{12} = \boxed{5,200,300}$.

(b) $\binom{25}{12}2^{12}(-3)^{13} = -\frac{25!}{12!13!}2^{12}3^{13} = \boxed{-33959763545702400}$.

(c) From (1.1), set $x = -1$ and $y = 1$, then we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (-1+1)^n = \boxed{0}$.

**Problem 9.** Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let $A$ denote the event $\{a, b\}$, and let $B$ denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-54]

***Solution***: Because the outcomes are equally likely, we can simply use classical probability.

(a) $P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{2}{5}}$

(b) $P(B) = \frac{|B|}{|\Omega|} = \boxed{\frac{3}{5}}$

(c) $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{5-2}{5} = \boxed{\frac{3}{5}}$

(d) $P(A \cup B) = \frac{|\{a,b,c,d,e\}|}{|\Omega|} = \frac{5}{5} = \boxed{1}$

(e) $P(A \cap B) = \frac{|\emptyset|}{|\Omega|} = \boxed{0}$

**Problem 10.** If $A$, $B$, and $C$ are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P\left((A \cup B) \cap C\right)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

***Solution***:

(a) Because $A$, $B$, and $C$ are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because $A$, $B$, and $C$ are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because $A$ and $B$ are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among $A$, $B$, and $C$, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$