ET 601: Computer Applications for Engineers **2013/2**

HW Solution 4 — Due: January 15

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Submit your work as one pdf file (which contains the solution for all the questions). The PDF file name should be ET601_HW4_FIRSTNAME.pdf in which the FIRSTNAME part is replaced by your first name.

Problem 1. Someone has rolled a fair dice twice. You know that one of the rolls turned up a face value of six. What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Not $\frac{1}{6}$.

Solution: Take as sample space the set $\{(i, j)|i, j = 1, \dots, 6\}$, where i and j denote the outcomes of the first and second rolls. A probability of $1/36$ is assigned to each element of the sample space. The event of two sixes is given by $A = \{(6, 6)\}$ and the event of at least one six is given by $B = (1, 6), \dots, (5, 6), (6, 6), (6, 5), \dots, (6, 1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 2. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = boxed 1/4$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1-2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 3. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 4. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.